# LECTURES ON SUPERGRAVITY p-BRANES \*

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We review the properties of classical p-brane solutions to supergravity theories, i.e. solutions that may be interpreted as Poincaré-invariant hyperplanes in spacetime. Topics covered include the distinction between elementary/electric and solitonic/magnetic solutions, examples of singularity and global structure, relations between mass densities, charge densities and the preservation of unbroken supersymmetry, diagonal and vertical Kaluza-Klein reduction families, Scherk-Schwarz reduction and domain walls, and the classification of multiplicities using duality symmetries.

## 1 Introduction

Supergravity theories originally arose from the desire to include supersymmetry into the framework of gravitational models, and this was in the hope that the resulting models might solve some of the outstanding difficulties of quantum gravity. One of these difficulties was the ultraviolet problem, on which early enthusiasm for supergravity's promise gave way to disenchantment when it became clear that local supersymmetry is not in fact sufficient to tame the notorious ultraviolet divergences.<sup>a</sup> Nonetheless, these theories won much admiration for their beautiful mathematical structure, which is due to the stringent constraints of their symmetries. These severely restrict the possible terms that can occur in the Lagrangian. For the maximal supergravity theories, there is simultaneously a great wealth of fields present and at the same time an impossibility of coupling any independent external field-theoretic "matter." It was only occasionally noticed in this early period that this impossibility of coupling to matter fields does not, however, rule out coupling to "relativistic objects" such as black holes, strings and membranes.

Indeed, a striking fact that has now been clearly recognized about supergravity theories is the degree to which they tell us precisely what kinds of external "matter" they will tolerate. The possibilities of such couplings may be learned in a fashion similar to the traditional derivation of the Schwarzshild solution in General Relativity, first searching for an isotropic solution in empty space, then considering later how this may be matched onto an interior matter

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<sup>&</sup>lt;sup>a</sup>For a review of ultraviolet behavior in supergravity theories, see Ref. <sup>1</sup>

source. In the case of supergravity theories, imposing the requirement that some part of the original theory's supersymmetry be left unbroken leads to the class of p-brane solutions that we shall review in this article. It is one of the marvels of the subject that this purely classical information from supergravity theories is now thought to be capable of yielding nonperturbative information on the superstring theories that we now see as the underlying quantum formulations of supergravity.

First, let us recall the way in which long-wavelength limits of string theories yield effective spacetime gravity theories. Consider, to begin with, the  $\sigma$ -model action <sup>11</sup> that describes a bosonic string moving in a background "condensate" of its own massless modes  $(g_{MN}, A_{MN}, \phi)$ :

$$I = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\gamma} \left[ \gamma^{ij} \partial_i x^M \partial_j x^N g_{MN}(x) + i\epsilon^{ij} \partial_i x^M \partial_j x^N A_{MN}(x) + \alpha' R(\gamma) \phi(x) \right]. \tag{1.1}$$

Every string theory contains a sector described by fields  $(g_{MN}, A_{MN}, \phi)$ ; these are the only fields that couple directly to the string worldsheet. In superstring theories, this sector is called the Neveu-Schwarz/Neveu-Schwarz (NS-NS) sector.

The  $\sigma$ -model action (1.1) is classically invariant under the worldsheet Weyl symmetry  $\gamma_{ij} \to \Lambda^2(z)\gamma_{ij}$ . Requiring cancellation of the anomalies in this symmetry at the quantum level gives differential-equation restrictions on the background fields  $(g_{MN}, A_{MN}, \phi)$  that may be viewed as effective equations of motion for these massless modes.<sup>12</sup> This system of effective equations may be summarized by the corresponding field-theory effective action

$$I_{\text{eff}} = \int d^{D}x \sqrt{-g} e^{-2\phi} \left[ (D - 26) - \frac{3}{2}\alpha'(R + 4\nabla^{2}\phi - 4(\nabla\phi)^{2} - \frac{1}{12}F_{MNP}F^{MNP} + \mathcal{O}(\alpha')^{2} \right], \qquad (1.2)$$

where  $F_{MNP} = \partial_M A_{NP} + \partial_N A_{PM} + \partial_P A_{MN}$  is the 3-form field strength for the  $A_{MN}$  gauge potential. The (D-26) term reflects the critical dimension for the bosonic string: flat space is a solution of the above effective theory only for D=26.

 $<sup>^</sup>b$ The present review is based largely upon Refs.<sup>2, 3, 4, 5, 6</sup> and focuses on classical solutions to supergravity theories without special regard to the structure of source terms that would reside in p-brane worldvolume actions. This restricted focus is made here for simplicity, as the structure of such p-brane worldvolume actions is still incompletely known. Earlier reviews of p-brane solutions including discussion of the worldvolume action sources may be found in Refs.<sup>7, 8, 9</sup> The occurrence of such solutions as D-brane backgrounds in string theory has been recently reviewed in Ref.<sup>10</sup>

The effective action for superstring theories contains a similar (NS-NS) sector, but with the substitution of (D-26) by (D-10), reflecting the different critical dimension for superstrings. In addition, superstring theories have a Ramond-Ramond (R-R) sector of further bosonic fields. For example, the type IIA theory <sup>14</sup> has R-R field strengths  $F_{[2]} = dA_{[1]}$  and  $F_{[4]} = dA_{[3]} + A_{[1]} \wedge F_{[3]}$ , where the [n] subscripts indicate the ranks of the forms. In the type IIB theory, <sup>15</sup> on the other hand, one has  $F_{[1]} = d\chi$ , where  $\chi$  is a R-R zero-form (i.e. a pseudoscalar field),  $F_{[3]}^{R} = dA_{[2]}^{R}$ , a second 3-form field strength making a pair together with  $F_{[3]}^{NS}$  from the NS-NS sector, and  $F_{[5]} = dA_{[4]}$ , which is a self-dual 5-form in D = 10,  $F_{[5]} = {}^*F_{[5]}$ .

Thus one naturally encounters field strengths of ranks 1–5 in type II theories. In addition, one may use  $\epsilon_{[10]}$  to dualize certain field strengths; e.g. the original  $F_{[3]}$  may be dualized to  ${}^*F_{[3]}$ , which is a 7-form. The upshot is that, in considering solutions to string-theory effective field equations, antisymmetric-tensor gauge field strengths of diverse ranks need to be taken into account. These field strengths will play an essential rôle in supporting the p-brane solutions that we shall describe.

The effective action (1.2) is written in the form directly obtained from string  $\sigma$ -model calculations. It is not written in the form generally preferred by relativists, which has a clean Einstein-Hilbert term free from exponential prefactors like  $e^{-2\phi}$ . One may rewrite the effective action in a different frame by making a Weyl-rescaling field redefinition  $g_{MN} \to e^{\lambda\phi}g_{MN}$ .  $I_{\text{eff}}$  as written in (1.2) is in the string frame; after an integration by parts, it takes the form

$$I^{\text{string}} = \int d^{10}x \sqrt{-g^{(s)}} e^{-2\phi} \left[ R(g^{(s)}) + 4\nabla_M \phi \nabla^M \phi - \frac{1}{12} F_{MNP} F^{MNP} \right] . \quad (1.3)$$

After making the transformation  $g_{MN}^{(e)} = e^{-\phi/2}g_{MN}^{(s)}$ , one obtains the *Einstein frame* action,

$$I^{\text{Einstein}} = \int d^{10}x \sqrt{-g^{(e)}} \Big[ R(g^{(e)}) - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{12} e^{\phi} F_{MNP} F^{MNP} \Big] , \quad (1.4)$$

where the indices are now raised and lowered with  $g_{MN}^{(e)}$ . To understand how this Weyl rescaling works, note that under x-independent rescalings, the connection  $\Gamma_{MN}^{\ P}$  is invariant. This carries over also to terms with  $\phi$  undifferentiated, which emerge from the  $e^{\lambda\phi}$  Weyl transformation. One then chooses  $\lambda$  so as to eliminate the  $e^{-2\phi}$  factor. Terms with  $\phi$  undifferentiated do change, however. As one can see in (1.4), the Weyl transformation is just what is needed to unmask the positive-energy sign of the kinetic term for the  $\phi$  field, despite the apparently negative sign of its kinetic term in  $I^{\text{string}}$ .

## 2 The p-brane ansatz

#### 2.1 General action and field equations

Motivated by the above summary of the effective field theories derived from string theories, let us now consider a classical system in D dimensions comprising the metric  $g_{MN}$ , a scalar field  $\phi$  and an (n-1)-form gauge potential  $A_{[n-1]}$  with corresponding field strength  $F_{[n]}$ , the whole described by the action

$$I = \int D^{D} x \sqrt{-g} \left[ R - \frac{1}{2} \nabla_{M} \phi \nabla^{M} \phi - \frac{1}{2n!} e^{a\phi} F_{[n]}^{2} \right]. \tag{2.1}$$

We shall consider later in more detail how (2.1) may be obtained by a consistent truncation from a full supergravity theory in D dimensions. The value of the parameter a controlling the interaction of the scalar field  $\phi$  with the field strength  $F_{[n]}$  in (2.1) will vary according to the cases considered in the following.

Varying the action (2.1) produces the following set of equations of motion:

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN} \tag{2.2a}$$

$$S_{MN} = \frac{1}{2(n-1)!} e^{a\phi} (F_M ... F_N ... - \frac{n-1}{n(D-2)} F^2 g_{MN})$$
 (2.2b)

$$\nabla_{M_1}(e^{a\phi}F^{M_1\cdots M_n}) = 0 ag{2.2c}$$

$$\Box \phi = \frac{a}{2n!} e^{a\phi} F^2 . \tag{2.2d}$$

#### 2.2 Electric and magnetic ansätze

In order to solve the above equations, we shall make a simplifying ansatz. We shall be looking for solutions preserving certain unbroken supersymmetries, and these will in turn require unbroken translational symmetries as well. For simplicity, we shall also require isotropic symmetry in the directions "transverse" to the translationally-symmetric ones. These restrictions can subsequently be relaxed in generalizations of the basic class of p-brane solutions that we shall discuss here. For this basic class of solutions, we make an ansatz requiring (Poincaré)<sub>d</sub> × SO(D-d) symmetry. One may view the sought-for solutions as flat d=p+1 dimensional hyperplanes embedded in the ambient D-dimensional spacetime; these hyperplanes may in turn be viewed as the histories, or worldvolumes, of p-dimensional spatial surfaces. Accordingly, let the spacetime coordinates be split into two ranges:  $x^M = (x^\mu, y^m)$ , where  $x^\mu$  ( $\mu = 0, 1, \cdots, p = d-1$ ) are coordinates adapted to the (Poincaré)<sub>d</sub> isometries

on the worldvolume and where  $y^m$   $(m=d,\cdots,D-1)$  are the coordinates "transverse" to the worldvolume.

An ansatz for the spacetime metric that respects the  $(Poincar\acute{e})_d \times SO(D-d)$  symmetry is <sup>13</sup>

$$ds^{2} = e^{2A(r)} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + e^{2B(r)} dy^{m} dy^{n} \delta_{mn} , \qquad (2.3)$$

where  $r = \sqrt{y^m y^m}$  is the isotropic radial coordinate in the transverse space. Since the metric components depend only on r, translational invariance in the worldvolume directions  $x^{\mu}$  and SO(D-d) symmetry in the transverse directions  $y^m$  is guaranteed.

The corresponding ansatz for the scalar field  $\phi(x^M)$  is simply  $\phi = \phi(r)$ .

For the antisymmetric tensor gauge field, we face a bifurcation of possibilities for the ansatz, the two possibilities being related by duality. The first possibility is naturally expressed directly in terms of the gauge potential  $A_{[n-1]}$ . Just as the Maxwell 1-form naturally couples to the worldline of a charged particle, so does  $A_{[n-1]}$  naturally couple to the worldvolume of a p=d-1=(n-1)-1 dimensional "charged" extended object. The "charge" here will be obtained from Gauss'-law surface integrals involving  $F_{[n]}$ , as we shall see later. Thus, the first possibility for  $A_{[n-1]}$  is to support a  $d_{\rm el}=n-1$  dimensional worldvolume. This is what we shall call the "elementary," or "electric" ansatz:

$$A_{\mu_1 \cdots \mu_{n-1}} = \epsilon_{\mu_1 \cdots \mu_{n-1}} e^{C(r)}$$
, others zero. (2.4)

SO(D-d) isotropicity and  $(Poincar\acute{e})_d$  symmetry are guaranteed here because the function C(r) depends only on the transverse radial coordinate r. Instead of the ansatz (2.4), expressed in terms of  $A_{[n-1]}$ , we could equivalently have given just the  $F_{[n]}$  field strength:

$$F_{m\mu_1\cdots\mu_{n-1}}^{(\mathrm{el})} = \epsilon_{\mu_1\cdots\mu_{n-1}}\partial_m e^{C(r)} \;, \quad \text{ others zero.} \eqno(2.5)$$

The worldvolume dimension for the elementary ansatz (2.4,2.5) is clearly  $d_{\rm el} = n - 1$ .

The second possible way to relate the rank n of  $F_{[n]}$  to the worldvolume dimension d of an extended object is suggested by considering the dualized field strength  ${}^*F$ , which is a (D-n) form. If one were to find an underlying gauge potential for  ${}^*F$  (locally possible by courtesy of a Bianchi identity), this would naturally couple to a  $d_{\rm so} = D - n - 1$  dimensional worldvolume. Since such a dualized potential would be nonlocally related to the fields appearing in the action (2.1), we shall not explicitly follow this construction, but shall

instead take this reference to the dualized theory as an easy way to identify the worldvolume dimension for the second type of ansatz. This "solitonic" or "magnetic" ansatz for the antisymmetric tensor field is most conveniently expressed in terms of the field strength  $F_{[n]}$ , which now has nonvanishing values only for indices corresponding to the transverse directions:

$$F_{m_1\cdots m_n}^{(\mathrm{mag})} = \lambda \epsilon_{m_1\cdots m_n p} \frac{y^p}{r^{n+1}}$$
, others zero, (2.6)

where the magnetic-charge parameter  $\lambda$  is a constant of integration, the only thing left undetermined by this ansatz. The power of r in the solitonic/magnetic ansatz is determined by requiring  $F_{[n]}$  to satisfy the Bianchi identity. Note that the worldvolume dimensions of the elementary and solitonic cases are related by  $d_{\rm so} = \tilde{d}_{\rm el} \equiv D - d_{\rm el} - 2$ ; note also that this relation is idempotent, i.e.  $(\tilde{d}) = d$ .

# 2.3 Curvature components and p-brane equations

In order to write out the field equations after insertion of the above ansätze, one needs to compute the Ricci tensor for the metric. This is most easily done by introducing veilbeins, *i.e.*, orthonormal frames,  $^{16}$  with tangent-space indices denoted by underlined indices:

$$g_{MN} = e_M^{\underline{E}} e_N^{\underline{F}} \eta_{EF} . \tag{2.7}$$

Next, one constructs the corresponding 1-forms:  $e^{\underline{E}} = dx^M e_M^{\underline{E}}$ . Splitting up the tangent-space indices  $\underline{E} = (\underline{\mu}, \underline{m})$  similarly to the world indices  $\underline{M} = (\mu, m)$ , we have for our ansätze the veilbein 1-forms

$$e^{\underline{\mu}} = e^{A(r)} dx^{\mu} , \qquad e^{\underline{m}} = e^{B(r)} dy^{m} . \qquad (2.8)$$

The corresponding spin connection 1-forms are determined by the condition that the torsion vanishes,  $de^{\underline{E}} + \omega^{\underline{E}}_{\underline{F}} \wedge e^{\underline{F}} = 0$ , which yields

$$\omega^{\underline{\mu}\underline{\nu}} = 0, \qquad \omega^{\underline{\mu}\underline{n}} = e^{-B(r)}\partial_n A(r)e^{\underline{\mu}} 
\omega^{\underline{m}\underline{n}} = e^{-B(r)}\partial_n B(r)e^{\underline{m}} - e^{-B(r)}\partial_m B(r)e^{\underline{n}}. \qquad (2.9)$$

The curvature 2-forms are then given by

$$R^{\underline{E}\,\underline{F}}_{[2]} = d\omega^{\underline{E}\,\underline{F}} + \omega^{\underline{E}\,\underline{D}} \wedge \omega_{\underline{D}}^{\,\underline{F}} \; . \tag{2.10}$$

<sup>&</sup>lt;sup>c</sup>Specifically, one finds  $\partial_q F_{m_1 \cdots m_n} = r^{-(n+1)} \left( \epsilon_{m_1 \cdots m_n q} - (n+1) \epsilon_{m_1 \cdots m_n p} y^p y_q / r^2 \right)$ ; upon taking the totally antisymmetrized combination  $[qm_1 \cdots m_n]$ , the factor of (n+1) is evened out between the two terms and then one finds from cycling a factor  $\sum_m y^m y_m = r^2$ , thus obtaining cancellation.

From the curvature components so obtained, one finds the Ricci tensor components

$$R_{\mu\nu} = -\eta_{\mu\nu}e^{2(A-B)}(A'' + d(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r}A')$$

$$R_{mn} = -\delta_{mn}(B'' + dA'B' + \tilde{d}(B')^2 + \frac{(2\tilde{d}+1)}{r}B' + \frac{d}{r}A') \qquad (2.11)$$

$$-\frac{y^m y^n}{r^2}(\tilde{B}'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r}B' - \frac{d}{r}A'),$$

where again,  $\tilde{d} = D - d - 2$ , and the primes indicate  $\partial/\partial r$  derivatives.

Substituting the above relations, one finds the set of equations that we need to solve to obtain the metric and  $\phi$ :

$$A'' + d(A')^{2} + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r}A' = \frac{\tilde{d}}{2(D-2)}S^{2} \qquad \{\mu\nu\}$$

$$B'' + dA'B' + \tilde{d}(B')^{2} + \frac{(2\tilde{d}+1)}{r}B' + \frac{d}{r}A' = -\frac{d}{2(D-2)}S^{2} \qquad \{\delta_{mn}\}$$

$$\tilde{B}'' + dA'' - 2dA'B' + d(A')^{2} - \tilde{d}(B')^{2}$$

$$-\frac{\tilde{d}}{r}B' - \frac{d}{r}A' + \frac{1}{2}(\phi')^{2} = \frac{1}{2}S^{2} \qquad \{y_{m}y_{n}\}$$

$$\phi'' + dA'\phi' + \tilde{d}B'\phi' + \frac{(d+1)}{r}\phi' = -\frac{1}{2}\varsigma aS^{2} , \qquad (2.12)$$

where  $\varsigma = \pm 1$  for the elementary/solitonic cases and the source appearing on the RHS of these equations is

$$S = \begin{cases} (e^{\frac{1}{2}a\phi - dA + C})C' & \text{elementary: } d = n - 1, \ \varsigma = +1 \\ \lambda(e^{\frac{1}{2}a\phi - \tilde{d}B})r^{-\tilde{d}-1} & \text{solitonic: } d = D - n - 1, \ \varsigma = -1. \end{cases}$$
(2.13)

# 2.4 p-brane solutions

The p-brane equations (2.12,2.13) are still rather daunting. In order to proceed further, we are going to take a hint from the requirements for supersymmetry preservation, which shall be justified in more detail later on. Accordingly, we shall now refine our ansätze by imposing the linearity condition

$$dA' + \tilde{d}B' = 0. (2.14)$$

After eliminating B using (2.14), the independent equations become <sup>3</sup>

$$\nabla^2 \phi = -\frac{1}{2} \varsigma a S^2 \tag{2.15a}$$

$$\nabla^2 A = \frac{\tilde{d}}{2(D-2)} S^2 \qquad (2.15b)$$

$$d(D-2)(A')^2 + \frac{1}{2}\tilde{d}(\phi')^2 = \frac{1}{2}\tilde{d}S^2$$
, (2.15c)

where, for spherically-symmetric (i.e. isotropic) functions in the transverse (D-d) dimensions, the Laplacian is  $\nabla^2 \phi = \phi'' + (\tilde{d}+1)r^{-1}\phi'$ .

Equations (2.15a,b) suggest that we now further refine the ansatze by imposing another linearity condition:

$$\phi' = \frac{-\varsigma a(D-2)}{\tilde{d}} A' \ . \tag{2.16}$$

At this stage, it is useful to introduce a new piece of notation, letting

$$a^2 = \Delta - \frac{2d\tilde{d}}{(D-2)} \ . \tag{2.17}$$

With this notation, equation (2.15c) gives

$$S^2 = \frac{\Delta(\phi')^2}{a^2} \,\,\,\,(2.18)$$

so that the remaining equation for  $\phi$  becomes  $\nabla^2 \phi + \frac{\varsigma \Delta}{2a} (\phi')^2 = 0$ , which can be re-expressed as a Laplace equation,<sup>d</sup>

$$\nabla^2 e^{\frac{\epsilon \Delta}{2a} \phi} = 0 . \tag{2.19}$$

Solving this in the transverse (D-d) dimensions with our assumption of transverse isotropicity (*i.e.* spherical symmetry) yields

$$e^{\frac{\zeta \Delta}{2a}\phi} = H(y) = 1 + \frac{k}{r^{\tilde{d}}}$$
  $k > 0$ , (2.20)

where the constant of integration  $\phi_{|r\to\infty}$  has been set equal to zero here for simplicity:  $\phi_{\infty}=0$ . The integration constant k in (2.20) sets the mass scale of the solution; it has been taken to be positive in order to ensure the absence of naked singularities at finite r. This positivity restriction is similar to the usual restriction to a positive mass parameter M in the standard Schwarzshild solution.

In the case of the elementary/electric ansatz, with  $\varsigma=+1$ , it still remains to find the function C(r) that determines the antisymmetric-tensor gauge field potential. In this case, it follows from (2.13) that  $S^2=e^{a\phi-1dA}(C'e^C)^2$ . Combining this with (2.15), one finds the relation

$$\frac{\partial}{\partial r}(e^C) = \frac{-\sqrt{\Delta}}{a}e^{-\frac{1}{2}a\phi + dA}\phi' \tag{2.21}$$

<sup>&</sup>lt;sup>d</sup>Note that Eq. (2.19) can also be more generally derived; for example, it still holds if one relaxes the assumption of isotropicity in the transverse space.

(where it should be remembered that a < 0). Finally, it is straightforward to verify that the relation (2.21) is consistent with the equation of motion for  $F_{[n]}$ :

$$\nabla^2 C + C'(C' + \tilde{B}' - dA' + a\phi') = 0.$$
 (2.22)

In order to simplify the explicit form of the solution, we now pick values of the integration constants to make  $A_{\infty} = B_{\infty} = 0$ , so that the solution tends to flat empty space at transverse infinity. Assembling the result, starting from the Laplace-equation solution H(y) (2.20), one finds <sup>8, 2</sup>

$$ds^{2} = H^{\frac{-4\bar{d}}{\Delta(D-2)}} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^{m} dy^{m}$$
 (2.23a)

$$e^{\phi} = H^{\frac{2a}{\varsigma\Delta}}$$
  $\varsigma = \begin{cases} +1, & \text{elementary/electric} \\ -1, & \text{solitonic/magnetic} \end{cases}$  (2.23b)

$$H(y) = 1 + \frac{k}{r^{\tilde{d}}},$$
 (2.23c)

and in the elementary/electric case, C(r) is given by

$$e^C = \frac{2}{\sqrt{\Delta}} H^{-1} \ . \tag{2.24}$$

In the solitonic/magnetic case, the constant of integration is related to the magnetic charge parameter  $\lambda$  in the ansatz (2.6) by

$$k = \frac{\sqrt{\Delta}}{2\tilde{d}}\lambda \ . \tag{2.25}$$

In the elementary/electric case, this relation may be taken to define the parameter  $\lambda$ .

#### 3 Examples

Consider now the bosonic sector of D = 11 supergravity, which has the action

$$I_{11} = \int d^{11}x \left\{ \sqrt{-g} \left(R - \frac{1}{48} F_{[4]}^2\right) + \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right\} . \tag{3.1}$$

There are two particular points to note about this action. The first is that no scalar field is present. This follows from the supermultiplet structure of the D=11 theory, in which all fields are gauge fields. In lower dimensions, of course, scalars do appear; e.g. the dilaton in D=10 type IIA supergravity emerges out of the D=11 metric upon dimensional reduction from D=11 to

D=10. The absence of the scalar that we had in our general discussion may be handled here simply by identifying the scalar coupling parameter a with zero, so that the scalar may be consistently truncated from our general action (2.1). Since  $a^2 = \Delta - 2d\tilde{d}/(D-2)$ , we identify  $\Delta = 2 \cdot 3 \cdot 6/9 = 4$  for the D=11 cases.

The second point to note is the presence of the FFA Chern-Simons term in (3.1). This term is required by D=11 local supersymmetry, with the coefficient as given in (3.1). Under the bosonic antisymmetric-tensor gauge transformation  $\delta A_{[3]} = d\Lambda_{[2]}$ , the FFA term in (3.1) is invariant (up to a total derivative) separately from the kinetic term.

In our general discussion given above in Sec. 2, we did not take into account the effects of such FFA terms. This omission, however, is not essential to the basic class of p-brane solutions that we are studying. Note that for n=4, the  $F_{[4]}$  antisymmetric tensor field strength supports either an elementary/electric solution with d=n-1=3 (i.e. a p=2 membrane) or a solitonic/magnetic solution with  $\tilde{d}=11-3-2=6$  (i.e. a p=5 brane). In both these elementary and solitonic cases, the FFA term in the action (3.1) vanishes and hence this term does not make any non-vanishing contribution to the metric field equations for our ansätze. For the antisymmetric tensor field equation, a further check is necessary, since there one requires the variation of the FFA term to vanish in order to consistently ignore it. The field equation for  $A_{[3]}$  is

$$\partial_{M} \left( \sqrt{-g} F^{MUVW} \right) + \frac{1}{2(4!)^{2}} \epsilon^{UVWx_{1}x_{2}x_{3}x_{4}y_{1}y_{2}y_{3}y_{4}} F_{x_{1}x_{2}x_{3}x_{4}} F_{y_{1}y_{2}y_{3}y_{4}} = 0 . \tag{3.2}$$

By direct inspection, one sees that the second term in this equation vanishes for both ansätze.

Next, we shall consider the elementary/electric and the solitonic/magnetic D=11 cases in detail. Subsequently, we shall explore how these particular solutions fit into wider, "black," families of p-branes.

# 3.1 D = 11 Elementary/electric 2-brane

From our general discussion in Sec. 2, we have the elementary-ansatz solution  $^{17}$ 

$$ds^{2} = (1 + \frac{k}{r^{6}})^{-\frac{2}{3}} d^{\mu} dx^{\nu} \eta_{\mu\nu} + (1 + \frac{k}{r^{6}})^{-\frac{1}{3}} dy^{m} dy^{m}$$

$$A_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda} (1 + \frac{k}{r^{6}})^{-1}, \quad \text{other components zero.}$$

$$\{\text{electric 2-brane: isotropic coordinates}\}$$

$$(3.3)$$

At first glance, this solution looks like it might be singular at r = 0. However, if one calculates the invariant components of the curvature tensor  $R_{MNPQ}$  and of the field strength  $F_{m\mu_1\mu_2\mu_3}$ , subsequently referred to an orthonormal frame by

introducing vielbeins as in (2.8), one finds these invariants to be nonsingular. Moreover, although the proper distance to the surface r=0 along a  $t=x^0=$  const. geodesic diverges, the surface r=0 can be reached along null geodesics in finite affine parameter.<sup>18</sup>

Thus, one may suspect that the metric as given in (3.3) does not in fact cover the entire spacetime, and so one should look for an analytic extension of it. Accordingly, one may consider a change to "Schwarzshild-type" coordinates by setting  $r = (\tilde{r}^6 - k)^{1/6}$ . The solution then becomes:<sup>18</sup>

$$ds^{2} = (1 + \frac{k}{r^{6}})^{2/3}(-dt^{2} + d\sigma^{2} + d\rho^{2}) + (1 + \frac{k}{r^{6}})^{-2}d\tilde{r}^{2} + \tilde{r}^{2}d\Omega_{7}^{2}$$

$$A_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda}(1 + \frac{k}{r^{6}}), \quad \text{other components zero,}$$

$$\{\text{electric 2-brane: Schwarzshild-type coordinates}\}$$

where we have supplied explicit worldvolume coordinates  $x^{\mu} = (t, \sigma, \rho)$  and where  $d\Omega_7^2$  is the line element on the unit 7-sphere, corresponding to the boundary  $\partial \mathcal{M}_{8T}$  of the 11-3=8 dimensional transverse space.

The Schwarzshild-like coordinates make the surface  $\tilde{r}=k^{1/6}$  (corresponding to r=0) look like a horizon. One may indeed verify that the normal to this surface is a *null* vector, confirming that  $\tilde{r}=k^{1/6}$  is in fact a horizon. This horizon is *degenerate*, however. Owing to the  $^2$ /3 exponent in the  $g_{00}$  component, curves along the t axis for  $\tilde{r} < k^{1/6}$  remain timelike, so that light cones do not "flip over" inside the horizon, unlike the situation for the classic Schwarzshild solution.

In order to see the structure of the membrane spacetime more clearly, let us change coordinates once again, setting  $\tilde{r}=k^{1/6}(1-R^3)^{-1/6}$ . Overall, the transformation from the original isotropic coordinates to these new ones is effected by setting  $\tilde{r}=k^{1/6}R^{1/2}/(1-R^3)^{1/6}$ . In these new coordinates, the solution becomes <sup>18</sup>

$$ds^{2} = \left\{ R^{2}(-dt^{2} + d\sigma^{2} + d\rho^{2}) + 4k^{\frac{1}{3}}R^{-2}dR^{2} \right\} + k^{\frac{1}{3}}d\Omega_{7}^{2}$$
(a)  
+  $k^{\frac{1}{3}}[(1 - R^{3})^{-\frac{1}{3}} - 1][4R^{-2}dR^{2} + d\Omega_{7}^{2}]$ (b)  
$$A_{\mu\nu\lambda} = R^{3}\epsilon_{\mu\nu\lambda} , \text{ other components zero.}$$
  
{electric 2-brane: interpolating coordinates}  
(3.5)

This form of the solution makes it clearer that the light-cones do not "flip over" in the region inside the horizon (which is now at R=0, with R<0 being the interior). The main usefulness of the third form (3.5) of the membrane solution, however, is that it reveals how the solution interpolates between other "vacuum" solutions of D=11 supergravity.<sup>18</sup> As  $R\to 1$ , the solution becomes flat, in the asymptotic exterior transverse region. As one approaches the horizon at R=0, line (b) of the metric in (3.5) vanishes

at least linearly in R. The residual metric, given in line (a), may then be recognized as a standard form of the metric on  $(AdS)_4 \times S^7$ , generalizing the Robinson-Bertotti solution on  $(AdS)_2 \times S^2$  in D=4. Thus, the membrane solution interpolates between flat space as  $R \to 1$  and  $(AdS)_4 \times S^7$  as  $R \to 0$  at the horizon.

Continuing on inside the horizon, one eventually encounters a true singularity at  $\tilde{r}=0$   $(R\to -\infty)$ . Unlike the singularity in the classic Schwarzshild solution, which is spacelike and hence unavoidable, the singularity in the membrane spacetime is timelike. Generically, geodesics do not intersect the singularity at a finite value of an affine parameter value. Radial null geodesics do intersect the singularity at finite affine parameter, however, so the spacetime is in fact genuinely singular. The timelike nature of this singularity, however, invites one to consider coupling a  $\delta$ -function source to the solution at  $\tilde{r}=0$ . Indeed, the D=11 supermembrane action, which generalizes the Nambu-Goto action for the string, is the unique "matter" system that can consistently couple to D=11 supergravity. Analysis of this coupling yields a relation between the parameter k in the solution (3.3) and the tension T of the supermembrane action:

$$k = \frac{\kappa^2 T}{3\Omega_7} \,, \tag{3.6}$$

where  $1/(2\kappa^2)$  is the coefficient of  $\sqrt{-g}R$  in the Einstein-Hilbert Lagrangian and  $\Omega_7$  is the volume of the unit 7-sphere  $\mathcal{S}^7$ , *i.e.* the solid angle subtended by the boundary at transverse infinity.

The global structure of the membrane spacetime  $^{18}$  is similar to the extreme Reissner-Nordstrom solution of General Relativity. This global structure is summarized by a Carter-Penrose diagram as shown in Figure 1, in which the angular coordinates on  $\mathcal{S}^7$  and also two ignorable worldsheet coordinates have been suppressed. As one can see, the region mapped by the isotropic coordinates does not cover the whole spacetime. This region, shaded in the diagram, is geodesically incomplete, since one may reach its boundaries  $\mathcal{H}^+$ ,  $\mathcal{H}^-$  along radial null geodesics at a finite affine-parameter value. These boundary surfaces are not singular, but, instead, constitute future and past horizons (one can see from the form (3.4) of the solution that the normals to these surfaces are null). The "throat"  $\mathcal{P}$  in the diagram should be thought of as an exceptional point at infinity, and not as a part of the central singularity.

The region exterior to the horizon interpolates between flat regions  $\mathcal{J}^{\pm}$  at future and past null infinities and a geometry that asymptotically tends to  $(AdS)_4 \times \mathcal{S}^7$  on the horizon. This interpolating portion of the spacetime, corresponding to the shaded region of Figure 1 which is covered by the isotropic coordinates, may be sketched as shown in Figure 2.

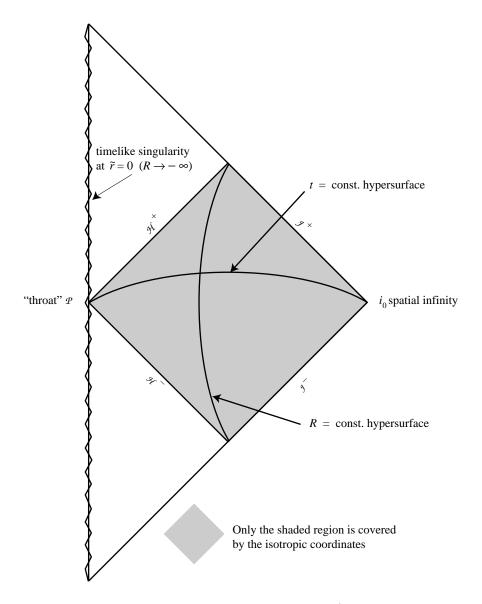


Figure 1: Carter-Penrose diagram for the D=11 elementary/electric 2-brane solution.

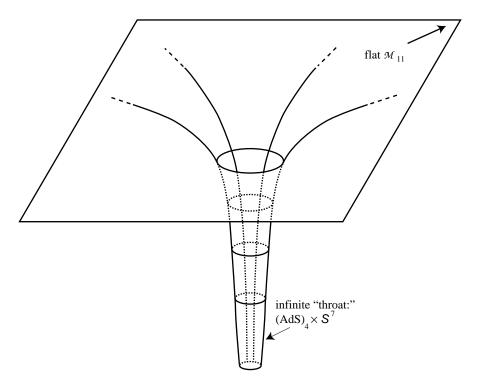


Figure 2: The D=11 elementary/electric 2-brane solution interpolates between flat space at  $\mathcal{J}^{\pm}$  and  $(\mathrm{AdS})_4 \times \mathcal{S}^7$  at the horizon.

# $3.2 \quad D = 11 \ Solitonic/magnetic 5-brane$

Now consider the 5-brane solution to the D=11 theory given by the solitonic ansatz for  $F_{[4]}$ . In isotropic coordinates, this solution is a magnetic 5-brane:<sup>22</sup>

$$ds^{2} = (1 + \frac{k}{r^{3}})^{-1/3} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + (1 + \frac{k}{r^{3}})^{2/3} dy^{m} dy^{m} \qquad \mu, \nu = 0, \cdots, 5$$

$$F_{m_{1} \cdots m_{4}} = 3k \epsilon_{m_{1} \cdots m_{4} p} \frac{y^{p}}{r^{5}} \qquad \text{other components zero.}$$

$$\{ \underline{\text{magnetic 5-brane: isotropic coordinates}} \}$$

$$(3.7)$$

As in the case of the elementary/electric membrane, this solution interpolates between two "vacua" of D=11 supergravity. Now, however, these asymptotic geometries consist of the flat region encountered as  $r\to\infty$  and of  $(\mathrm{AdS})_7\times\mathcal{S}^4$  as one approaches r=0, which once again is a degenerate horizon. Combining two coordinate changes analogous to those of the elementary case,

 $r = (\tilde{r}^3 - k)^{1/3}$  and  $\tilde{r} = k^{1/3}(1 - R^6)^{-1/3}$ , one has an overall transformation

$$r = \frac{k^{1/3}R^2}{(1 - R^6)^{1/3}} \ . \tag{3.8}$$

After these coordinate changes, the metric becomes

$$ds^{2} = R^{2} dx^{\mu} dx^{\nu} \eta_{\mu\nu} + k^{2/3} \left[ \frac{4R^{-2}(1+R^{6})^{2}}{(1-R^{6})^{8/3}} dR^{2} + \frac{d\Omega_{4}^{2}}{(1-R^{6})^{2/3}} \right] .$$
{magnetic 5-brane: interpolating coordinates} (3.9)

Once again, the surface  $r = 0 \leftrightarrow R = 0$  may be seen from (3.9) to be a nonsingular degenerate horizon. In this case, however, not only do the light cones maintain their timelike orientation when crossing the horizon, as already happened in the electric case (3.5), but now the magnetic solution (3.9) is in fact fully  $symmetric^{23}$  under a discrete isometry  $R \to -R$ .

Given this isometry  $R \to -R$ , one can identify the spacetime region  $R \leq 0$ with the region R > 0. This identification is analogous to the identification one naturally makes for flat space when written in polar coordinates, with the metric  $ds_{\text{flat}}^2 = -dt^2 + dr^2 + r^2d^2$ . Ostensibly, in these coordinates there appear to be separate regions of flat space with  $r \gtrsim 0$ , but, owing to the existence of the isometry  $r \to -r$ , these regions may be identified. Accordingly, in the solitonic/magnetic 5-brane spacetime, we identify the region  $-1 < R \le 0$  with the region  $0 \le R < 1$ . In the asymptotic limit where  $R \to -1$ , one finds an asymptotically flat geometry that is indistinguishable from the region where  $R \to +1$ , i.e. where  $r \to \infty$ . Thus, there is no singularity at all in the solitonic/magnetic 5-brane geometry. There is still an infinite "throat," however, at the horizon, and the region covered by the isotropic coordinates might again be sketched as in Figure 2, except now with the asymptotic geometry down the "throat" being  $(AdS)_7 \times S^4$  instead of  $(AdS)_4 \times S^7$  as for the elementary/electric solution. The Carter-Penrose diagram for the solitonic/magnetic 5-brane solution is given in Figure 3, where the full diagram extends indefinitely by "tiling" the section shown. Upon using the  $R \to -R$  isometry to make discrete identifications, however, the whole of the spacetime may be considered to consist of just region I, which is the region covered by the isotropic coordinates (3.7).

After identification of the  $R \geq 0$  regions, the 5-brane spacetime (3.7) is geodesically complete. Unlike the case of the elementary membrane solution (3.3,3.5), one finds in the solitonic/magnetic case that the null geodesics passing through the horizon at R=0 continue to evolve in their affine parameters without bound as  $R \to -1$ . Thus, the solitonic 5-brane solution is completely non-singular.

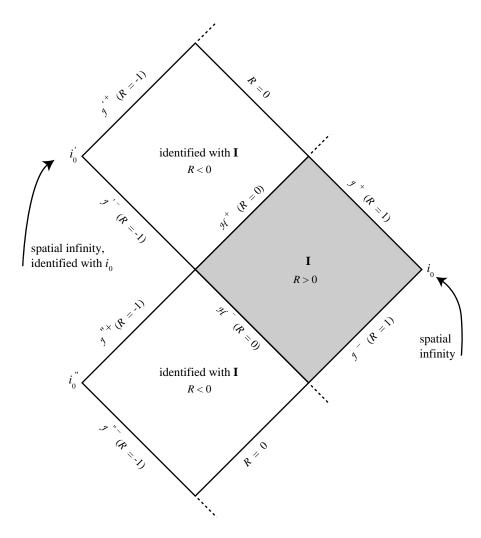


Figure 3: Carter-Penrose diagram for the solitonic/magnetic 5-brane solution.

The electric and magnetic D=11 solutions discussed here and in the previous subsection are special in that they do not involve a scalar field, since the bosonic sector of D=11 supergravity (3.1) does not even contain a scalar field. Similar solutions occur in other situations where the parameter a (2.17) for a field strength supporting a p-brane solution vanishes, in which cases the scalar fields may consistently be set to zero; this happens for (D,d)=(11,3), (11,5), (10,4), (6,2), (5,1), (5,2) and (4,1). In these special cases, the solutions are nonsingular at the horizon and so one may analytically continue through to the other side of the horizon. When d is even for "scalarless" solutions of this type, there exists a discrete isometry analogous to the  $R \to -R$  isometry of the D=11 5-brane solution (3.9), allowing the outer and inner regions to be identified.<sup>23</sup> When d is odd in such cases, the analytically-extended metric eventually reaches a timelike curvature singularity at  $\tilde{r}=0$ .

When  $a \neq 0$  and the scalar field associated to the field strength supporting a solution cannot be consistently set to zero, then the solution is singular at the horizon, as can be seen directly in the scalar solution (2.20) itself (where we recall that in isotropic coordinates, the horizon occurs at r = 0)

#### 3.3 Black branes

In order to understand better the family of supergravity solutions that we have been discussing, let us now consider a generalization that lifts the degenerate nature of the horizon. Written in Schwarzshild-type coordinates, one finds the generalized "black brane" solution  $^{24,25}$ 

$$ds^{2} = -\frac{\sum_{+}}{\sum_{-}^{\left\{1 - \frac{4\tilde{d}}{\Delta(D-2)}\right\}}} dt^{2} + \sum_{-}^{\frac{4\tilde{d}}{\Delta(D-2)}} dx^{i} dx^{i}$$

$$+ \frac{\sum_{-}^{\left\{\frac{2a^{2}}{\Delta\tilde{d}} - 1\right\}}}{\sum_{+}} d\tilde{r}^{2} + \tilde{r}^{2} \sum_{-}^{\frac{2a^{2}}{\Delta\tilde{d}}} d\Omega_{D-d-1}^{2}$$

$$e^{\frac{\varsigma\Delta}{2a}\phi} = \sum_{-}^{-1} \qquad \Sigma_{\pm} = 1 - \left(\frac{r_{\pm}}{\tilde{r}}\right)^{\tilde{d}}.$$
{black brane: Schwarzshild-type coordinates}

The antisymmetric tensor field strength for this solution corresponds to a charge parameter  $\lambda = 2\tilde{d}/\sqrt{\Delta}(r_=r_-)^{\tilde{d}/2}$ , either electric or magnetic.

The characteristic feature of the above "blackened" p-branes is that they have a nondegenerate, nonsingular outer horizon at  $\tilde{r}=r_+$ , at which the light cones "flip over." At  $\tilde{r}=r_-$ , one encounters an inner horizon, which, however, coincides in general with a curvature singularity. The singular nature of the solution at  $\tilde{r}=r_-$  is apparent in the scalar  $\phi$  in (3.10). For solutions with

 $p \geq 1$ , the singularity at the inner horizon persists even in cases where the scalar  $\phi$  is absent.

The extremal limit of the black brane solution occurs for  $r_+ = r_-$ . When a = 0 and scalars may consistently be set to zero, the singularity at the horizon  $r_+ = r_-$  disappears and then one may analytically continue through the horizon. In this case, the light cones do not "flip over" at the horizon because one is really crossing two coalesced horizons, and the coincident "flips" of the light cones cancel out.

The generally singular nature of the inner horizon of the non-extreme solution (3.10) shows that the "location" of the p-brane in spacetime should normally be thought to coincide with the inner horizon, or with the degenerate horizon in the extremal case.

#### 4 Masses, Charges and Supersymmetry

The p-brane solutions that we have been studying are supported by antisymmetric tensor gauge field strengths that fall off at transverse infinity like  $r^{-(\tilde{d}+1)}$ , as one can see from (2.5,2.24,2.6). This asymptotic falloff is slow enough to give a nonvanishing total charge density from a Gauss' law flux integral at transverse infinity, and we shall see that, for the "extremal" class of solutions that is our main focus, the mass density of the solution saturates a "Bogomol'ny bound" with respect to the charge density. This relation between densities is in turn connected to another feature of these solutions: although purely bosonic, they preserve unbroken some portion of the original supersymmetry of the corresponding supergravity theory.\(^{13}

#### 4.1 Masses

Let us begin with the mass density. Since the p-brane solutions have translational symmetry in their p spatial worldvolume directions, the total energy as measured by a surface integral at spatial infinity diverges, owing to the infinite extent. What is thus more appropriate to consider instead is the value of the density, energy/(unit p-volume). Since we are considering solutions in their rest frames, this will also give the value of mass/(unit p-volume), or tension of the solution. Instead of the standard spatial  $d^{D-2}\Sigma^a$  surface integral, this will be a  $d^{(D-d-1)}\Sigma^m$  surface integral over the boundary  $\partial \mathcal{M}_{\rm T}$  of the transverse space.

The ADM formula for the energy density written as a Gauss'-law integral

(see, e.g., Ref. 16) is, dropping the divergent spatial  $d\Sigma^{\mu=i}$  integral,

$$\mathcal{E} = \frac{1}{4\Omega_{D-d-1}} \int_{\partial \mathcal{M}_{T}} d^{D-d-1} \Sigma^{m} (\partial^{n} h_{mn} - \partial_{m} h_{b}^{b}) , \qquad (4.1)$$

written for  $g_{MN} = \eta_{MN} + h_{MN}$  tending asymptotically to flat space in Cartesian coordinates, and with a, b spatial indices running over the values  $\mu = i = 1, \ldots, d-1$ ;  $m = d, \ldots, D-1$ .  $\Omega_{D-d-1}$  is the volume of the  $\mathcal{S}^{D-d-1}$  unit sphere. For the general p-brane solution (2.23), one finds

$$h_{mn} = \frac{4kd}{\Delta(D-2)r\tilde{d}}\delta_{mn} , \qquad h_b^b = \frac{8k(d+\frac{1}{2}\tilde{d})}{\Delta(D-2)r\tilde{d}} , \qquad (4.2)$$

and, since  $d^{(D-d-1)}\Sigma^m = r^{\tilde{d}}y^m d\Omega^{(D-d-1)}$ , one finds

$$\mathcal{E} = \frac{k\tilde{d}}{\Lambda} \,, \tag{4.3}$$

and, recalling that  $k = \sqrt{\Delta \lambda}/(2\tilde{d})$ , we consequently have a relation between the mass per unit p volume and the charge parameter of the solution

$$\mathcal{E} = \frac{\lambda}{2\sqrt{\Delta}} \ . \tag{4.4}$$

By contrast, the black brane solution (3.10) has  $\mathcal{E} > \lambda/(2\sqrt{\Delta})$ , so the extremal *p*-brane solution (2.23) is seen to *saturate* the inequality  $\mathcal{E} \geq \lambda/(2\sqrt{\Delta})$ .

#### 4.2 Charges

As one can see from (4.3,4.4), the relation (2.25) between the integration constant k in the solution (2.23) and the charge parameter  $\lambda$  implies a deep link between the energy density and certain electric or magnetic charges. In the electric case, this charge is a quantity conserved by virtue of the equations of motion for the antisymmetric tensor gauge field  $A_{[n-1]}$ , and has generally become known as a "Page charge," after its first discussion in Ref.<sup>26</sup> To be specific, if we once again consider the bosonic sector of D=11 supergravity theory (3.1), for which the antisymmetric tensor field equation was given in (3.2), one finds the Gauss'-law form conserved quantity <sup>26</sup>

$$U = \frac{1}{4\Omega_7} \int_{\partial \mathcal{M}_8} (*F_{[4]} + \frac{1}{2} A_{[3]} \wedge F_{[4]}) , \qquad \{\underline{\text{electric charge}}\}$$
 (4.5)

where the integral of the 7-form integrand is over the boundary at infinity of an arbitrary infinite 8-dimensional spacelike subspace of D=11 spacetime. This arbitrariness of choice in the 8-dimensional spacelike subspace means that (4.5) in fact represents a whole set of conserved charges. A basis of these charges may be obtained by taking the embedding of  $\mathcal{M}_8$  into the 10-dimensional spatial hypersurface to be specified by a volume-element 2-form. Accordingly, the electric Page charge (4.5) should properly be denoted by  $U_{AB}$ .

For the p-brane solutions (2.23), the  $\int A \wedge F$  term in (4.5) vanishes. The  $\int {}^*F$  term does, however, give a contribution in the elementary/electric case, provided one picks  $\mathcal{M}_8$  to be the transverse space to the d=3 membrane worldvolume,  $\mathcal{M}_{8T}$ . The surface element for this transverse space is  $d\Sigma^m_{(7)}$ , so for the p=2 elementary membrane solution (3.3), one finds

$$U = \frac{1}{4\Omega_7} \int_{\partial M_{2T}} d\Sigma_{(7)}^m F_{m012} = \frac{\lambda}{4} . \tag{4.6}$$

Since the D=11  $F_{[4]}$  field strength supporting this solution has  $\Delta=4$ , the mass/charge relation is

$$\mathcal{E} = U = \frac{\lambda}{4} \ . \tag{4.7}$$

Thus, like the classic extreme Reissner-Nordstrom black-hole solution to which it is strongly related (as can be seen from the Carter-Penrose diagram given in Figure 1), the D=11 membrane solution has equal mass and charge densities, saturating the inequality  $\mathcal{E} \geq U$ .

Given the existence of an electric-type charge (4.5), one also expects to find a magnetic-type charge, which, however, should be conserved topologically, i.e. by virtue of the Bianchi identity  $dF_{[4]} = 0$ . This magnetic-type charge, being an integral over a four-form  $F_{[4]}$ , necessarily again involves integration over a submanifold of the spatial hypersurface of D = 11 spacetime:

$$V = \frac{1}{4\Omega_4} \int_{\partial \mathcal{M}_{\pi}} F_{[4]} ; \qquad \{\underline{\text{magnetic charge}}\}$$
 (4.8)

the surface integral now being taken over the boundary at infinity of a spacelike 5-dimensional subspace. As with the electric-type Page charge  $U_{AB}$ , V really represents a whole multiplet of charges, depending on the embedding of the subsurface  $\mathcal{M}_5$  into the 10-dimensional spatial hypersurface. This embedding may be specified in terms a volume 5-form, so the magnetic charge should properly be denoted by a 5-form  $V_{ABCDE}$ .

It is the magnetic form of charge (4.8) that is carried by the solitonic/magnetic 5-brane solution (3.7). Once again, there is only one orientation of the

subsurface  $\mathcal{M}_5$  that gives a nonvanishing contribution, *i.e.* that with  $\mathcal{M}_5 = \mathcal{M}_{5T}$ , the transverse space to the d = 6 worldvolume:

$$V = \frac{1}{4\Omega_4} \int_{\partial \mathcal{M}_{5T}} d\Sigma_{(4)}^m \epsilon_{mnpqr} F^{npqr} = \frac{\lambda}{4} . \tag{4.9}$$

Thus, in the solitonic/magnetic 5-brane case as well, we have a saturation of the mass-charge inequality:

$$\mathcal{E} = V = \frac{\lambda}{4} \ . \tag{4.10}$$

# 4.3 Supersymmetry

Since the bosonic solutions that we have been considering are consistent truncations of D=11 supergravity, they must also possess another conserved quantity, the supercharge. Admittedly, since the supercharge is a Grassmanian (anticommuting) quantity, its value will clearly be zero for the class of purely bosonic solutions that we have been discussing. However, the functional form of the supercharge is still important, as it determines the form of the asymptotic supersymmetry algebra. The Gauss'-law form of the supercharge is given as an integral over the boundary of the spatial hypersurface. For the D=11 solutions, this surface of integration is the boundary at infinity  $\partial \mathcal{M}_{10}$  of the D=10 spatial hypersurface; the supercharge is then  $^{27}$ 

$$Q = \int_{\partial \mathcal{M}_{10}} \Gamma^{0bc} \psi_c d\Sigma_{(9)b} . \tag{4.11}$$

One can also rewrite this in fully Lorentz-covariant form, where  $d\Sigma_{(9)b} = d\Sigma_{(9)0b} \to d\Sigma_{(9)AB}$ :

$$Q = \int_{\partial \mathcal{M}_{10}} \Gamma^{ABC} \psi_C d\Sigma_{(9)AB} . \tag{4.12}$$

After appropriate definitions of Poisson brackets, the D=11 supersymmetry algebra for the supercharge (4.11,4.12) is found to be  $^{28}$ 

$$\{Q,Q\} = C(\Gamma^A P_A + \Gamma^{AB} U_{AB} + \Gamma^{ABCDE} V_{ABCDE}), \qquad (4.13)$$

where C is the charge conjugation matrix,  $P_A$  is the energy-momentum 11-vector and  $U_{AB}$  and  $V_{ABCDE}$  are electric- and magnetic-type charges of precisely the sorts discussed in the previous subsection. Thus, the supersymmetry algebra wraps together all of the conserved Gauss'-law type quantities that we have discussed.

The positivity of the  $Q^2$  operator on the LHS of the algebra (4.13) is at the root of the  $Bogomol'ny\ bounds^{29,\,23,\,30}$ 

$$\mathcal{E} \geq (2/\sqrt{\Delta})U$$
 {electric} (4.14a)

$$\mathcal{E} \geq (2/\sqrt{\Delta})V$$
 {magnetic} (4.14b)

that are saturated by the p-brane solutions.

The saturation of the Bogomol'ny inequalities by the p-brane solutions is an indication that they fit into special types of supermultiplets. All of these bound-saturating solutions share the important property that they leave some portion of the supersymmetry unbroken. Within the family of p-brane solutions that we have been discussing, it turns out <sup>30</sup> that the  $\Delta$  values of such "supersymmetric" p-branes are of the form  $\Delta = 4/N$ , where N is the number of antisymmetric tensor field strengths participating in the solution (distinct, but of the same rank). The different charge contributions to the supersymmetry algebra occurring for different values of N (hence different  $\Delta$ ) affect the Bogomol'ny bounds as shown in (4.14).

In order to see how a purely bosonic solution may leave some portion of the supersymmetry unbroken, consider specifically again the membrane solution of D = 11 supergravity.<sup>17</sup> This theory<sup>27</sup> has just one spinor field, the gravitino  $\psi_M$ . Checking for the consistency of setting  $\psi_M = 0$  with the supposition of some residual supersymmetry with parameter  $\epsilon(x)$  requires solving the equation

$$\delta \psi_{A_{|\psi=0}} = \tilde{D}_A \epsilon = 0 , \qquad (4.15)$$

where  $\psi_A = e_A{}^M \psi_M$  and

$$\tilde{D}_{A}\epsilon = D_{A}\epsilon - \frac{1}{288} \left( \Gamma_{A}^{BCDE} - 8\delta_{A}^{B} \Gamma^{CDE} \right) F_{BCDE}\epsilon 
D_{A}\epsilon = \left( \partial_{A} + \frac{1}{4} \omega_{A}^{BC} \Gamma_{BC} \right) \epsilon .$$
(4.16)

Solving the equation  $\tilde{D}_A \epsilon = 0$  amounts to finding a Killing spinor field in the presence of the bosonic background. Since the Killing spinor equation (4.15) is linear in  $\epsilon(x)$ , the Grassmanian (anticommuting) character of this parameter is irrelevant to the problem at hand, which thus reduces effectively to solving (4.15) for a commuting quantity.

In order to solve the Killing spinor equation (4.15) in a p-brane background, it is convenient to adopt an appropriate basis for the  $D=11~\Gamma$  matrices. For the d=3 membrane background, one would like to preserve  $\mathrm{SO}(2,1)\times\mathrm{SO}(8)$  covariance. An appropriate basis that does this is

$$\Gamma_A = (\gamma_\mu \otimes \Sigma_9, \mathbb{1}_{(2)} \otimes \Sigma_m) , \qquad (4.17)$$

where  $\gamma_{\mu}$  and  $\mathbb{1}_{(2)}$  are  $2 \times 2$  SO(2,1) matrices;  $\Sigma_{9}$  and  $\Sigma_{m}$  are  $16 \times 16$  SO(8) matrices, with  $\Sigma_{9} = \Sigma_{3}\Sigma_{4} \dots \Sigma_{10}$ , so  $\Sigma_{9}^{2} = \mathbb{1}_{(16)}$ . The most general spinor field consistent with (Poincaré)<sub>3</sub> × SO(8) invariance in this spinor basis is of the form

$$\epsilon(x,y) = \epsilon_2 \otimes \eta(r) , \qquad (4.18)$$

where  $\epsilon_2$  is a constant SO(2, 1) spinor and  $\eta(r)$  is an SO(8) spinor depending only on the isotropic radial coordinate r;  $\eta$  may be further decomposed into  $\Sigma_9$  eigenstates by the use of  $\frac{1}{2}(\mathbb{1} \pm \Sigma_9)$  projectors.

Analysis of the Killing spinor condition (4.15) in the above spinor basis leads to the following requirements on the background and on the spinor field  $\eta(r)$ :<sup>13, 17</sup>

- 1. The background must satisfy the conditions 3A' + 6B' = 0 and  $C'e^C = 3A'e^{3A}$ . The first of these conditions is, however, precisely the linearity-condition refinement (2.14) that we made in the *p*-brane ansatz; the second condition follows from the ansatz refinement (2.16) (considered as a condition on  $\phi'/a$ ) and from (2.21). Thus, what appeared previously to be simplifying specializations in the derivation given in Section 2 turn out in fact to be conditions required for supersymmetric solutions.
- 2.  $\eta(r) = e^{-C(r)/6}\eta_0$ , where  $\eta_0$  is a constant SO(8) spinor. Note that, after imposing this requirement, at most a finite number of parameters can remain unfixed in the product spinor  $\epsilon\eta_0$ ; *i.e.* the *local* supersymmetry of the D=11 theory is almost entirely broken by any particular solution. The maximum number of rigid unbroken supersymmetry components is achieved for D=11 flat space, which has a full 32-component rigid supersymmetry.
- 3.  $(1 \Sigma_9)\eta_0 = 0$ , so the constant SO(8) spinor  $\eta_0$  is also required to be *chiral*<sup>e</sup>. This cuts the number of surviving parameters in the product  $\epsilon\eta_0$  by *half*: the total number of surviving rigid supersymmetries in  $\epsilon(x,y)$  is thus  $2 \cdot 8 = 16$  (real spinor components). Since this is half of the maximum possible number (*i.e.* half of that for flat space), one says that the membrane solution "preserves half" of the supersymmetry.

Similar consideration of the solitonic/magnetic 5-brane solution <sup>22</sup> (3.7) shows that it also preserves half the supersymmetry in the above sense. Half preservation is the maximum that can be achieved short of an empty-space

<sup>&</sup>lt;sup>e</sup>The specific chirality indicated here is correlated with the sign choice made in the elementary/electric form ansatz (2.4); one may accordingly observe from (3.1) that a D=11 parity transformation requires a sign flip of  $A_{[3]}$ .

solution, and when this happens, it corresponds to the existence of zero eigenvalues of the operator  $\{Q,Q\}$ . The positive semi-definiteness of this operator is the underlying principle in the derivation of the Bogomol'ny bounds  $(4.14)^{29,23,30}$  A consequence of this positive semi-definiteness is that zero eigenvalues correspond to solutions that saturate the Bogomol'ny inequalities (4.14), and these solutions preserve one component of unbroken supersymmetry for each such zero eigenvalue.

#### 5 Kaluza-Klein dimensional reduction

Let us return now to the arena of purely bosonic field theories, and consider the relations between various bosonic-sector theories and the corresponding relations between p-brane solutions. It is well-known that supergravity theories are related by dimensional reduction from a set of basic theories, the largest of which being D=11 supergravity. The spinor sectors of the theories are equally well related by dimensional reduction, but in the following, we shall restrict our attention to the purely bosonic sector.

In order to set up the procedure, let us consider a theory in (D+1) dimensions, but break up the metric in D-dimensionally covariant pieces:

$$d\hat{s}^2 = e^{2\alpha\varphi}ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_M dx^M)^2$$
(5.1)

where carets denote (D+1)-dimensional quantities corresponding to the (D+1)-dimensional coordinates  $x^{\hat{M}}=(x^M,z);\ ds^2$  is the line element in D dimensions and  $\alpha$  and  $\beta$  are constants. The D-scalar  $\varphi$  emerges from (D+1) dimensions as  $(2\beta)^{-1} \ln g_{zz}$ . Adjustment of the constants  $\alpha$  and  $\beta$  is necessary to obtain desired structures in D dimensions. In particular, one should pick  $\beta = -(D-2)\alpha$  in order to have the Einstein-frame form of the gravitational action in (D+1) dimensions go over to the Einstein-frame form of the action in D dimensions.

The essential step in a Kaluza-Klein dimensional reduction is a consistent truncation of the field variables, generally made by choosing them to be independent of the reduction coordinate z. By consistent truncation, we always mean a restriction on the variables that commutes with variation of the action to produce the field equations, *i.e.* a restriction such that solutions to the equations for the restricted variables are also solutions to the equations for the unrestricted variables. This ensures that the lower-dimensional solutions that we shall obtain are also particular solutions to higher-dimensional supergravity equations as well. Making the parameter choice  $\beta = -(D-2)\alpha$  to preserve

the Einstein-frame form of the action, one obtains

$$\sqrt{-\hat{g}}R\Big(\hat{g}) = \sqrt{-g}(R(g) - (D-1)(D-2)\alpha^2 \nabla_M \varphi \nabla^M \varphi - \frac{1}{4}e^{-2(D-1)\alpha\varphi} \mathcal{F}_{MN} \mathcal{F}^{MN}\Big)$$
(5.2)

where  $\mathcal{F} = d\mathcal{A}$ . If one now chooses  $\alpha^2 = [2(D-1)(D-2)]^{-1}$ , the  $\varphi$  kinetic term becomes conventionally normalized.

Next, one needs to establish the reduction ansatz for the (D+1)-dimensional antisymmetric tensor gauge field  $\hat{F}_{[n]} = d\hat{A}_{[n-1]}$ . Clearly, among the n-1 antisymmetrized indices of  $\hat{A}_{[n-1]}$  at most one can take the value z, so we have the decomposition

$$\hat{A}_{[n-1]} = B_{[n-1]} + B_{[n-2]} \wedge dz . \tag{5.3}$$

All of these reduced fields are to be taken to be functionally independent of z. For the corresponding field strengths, first define

$$G_{[n]} = dB_{[n-1]} \tag{5.4a}$$

$$G_{[n-1]} = dB_{[n-2]}.$$
 (5.4b)

However, these are not exactly the most convenient quantities to work with, since a certain "Chern-Simons" structure appears upon dimensional reduction. The metric in (D+1) dimensions couples to all fields, and, consequently, dimensional reduction will produce some terms with undifferentiated Kaluza-Klein vector fields  $\mathcal{A}_{M}$  coupling to D-dimensional antisymmetric tensors. Accordingly, it is useful to introduce

$$G'_{[n]} = G_{[n]} - G_{[n-1]} \wedge \mathcal{A} ,$$
 (5.5)

where the second term in (5.5) may be viewed as a Chern-Simons correction from the reduced D-dimensional point of view.

At this stage, we are ready to perform the dimensional reduction of our general action (2.1). We find

$$\hat{I} =: \int d^{D+1}x \sqrt{-\hat{g}} \left[ R(\hat{g}) - \frac{1}{2} \nabla_{\hat{M}} \phi \nabla^{\hat{M}} \phi - \frac{1}{2n!} e^{\hat{a}\phi} \hat{F}_{n]}^{2} \right]$$
 (5.6)

reduces to

$$I = \int d^{D}x \sqrt{-g} \left[ R - \frac{1}{2} \nabla_{M} \phi \nabla^{M} \phi - \frac{1}{2} \nabla_{M} \varphi \nabla^{M} \varphi - \frac{1}{4} e^{-2(D-1)\alpha \varphi} \mathcal{F}_{[2]}^{2} \right]$$
$$- \frac{1}{2n!} e^{-2(n-1)\alpha \varphi + \hat{\alpha} \phi} G_{[n]}^{\prime 2} - \frac{1}{2(n-1)!} e^{2(D-n)\alpha \varphi + \hat{\alpha} \phi} G_{[n-1]}^{2} \right]. (5.7)$$

Although the dimensional reduction (5.7) has produced a somewhat complicated result, the important point to note is that each of the D-dimensional antisymmetric-tensor field strength terms  $G'^2_{[n]}$  and  $G^2_{[n-1]}$  has an exponential prefactor of the form  $e^{a_r\tilde{\phi}_r}$ , where the  $\tilde{\phi}_r$ , r=(n,n-1) are SO(2)-rotated combinations of  $\varphi$  and  $\phi$ . Now, keeping just one while setting to zero the other two of the three gauge fields  $(\mathcal{A}_{[1]}, B_{[n-2]}, B_{n-1]})$ , but retaining at the same time the scalar-field combination appearing in the corresponding exponential prefactor, is a consistent truncation. Thus, any one of the three field strengths  $(\mathcal{F}_{[2]}, G_{[n-1]}, G'_{[n]})$ , retained alone together with its corresponding scalar-field combination, can support p-brane solutions in D dimensions of the form that we have been discussing.

An important point to note here is that in each of the  $e^{a_r\tilde{\phi}}$  prefactors, the coefficient  $a_r$  satisfies

$$a_r^2 = \Delta - \frac{2d_r\tilde{d}_r}{(D-2)} = \Delta - \frac{2(r-1)(D-r-1)}{(D-2)}$$
 (5.8)

with the same value of  $\Delta$  as for the "parent" coupling parameter  $\hat{a}$ , satisfying

$$\hat{a}_r^2 = \Delta - \frac{2d_{(n)}\tilde{d}_{(n)}}{((D+1)-2)} = \Delta - \frac{2(n-1)(D-n)}{(D-1)}$$
(5.9)

in D+1 dimensions. Thus, although the individual parameters  $a_r$  are both D-and r-dependent, the quantity  $\Delta$  is preserved under Kaluza-Klein reduction for both of the "descendant" field-strength couplings (to  $G'^2_{[n]}$  or to  $G^2_{[n-1]}$ ) coming from the original term  $e^{\hat{a}\phi}\hat{F}^2_{[n]}$ . The 2-form field strength  $\mathcal{F}_{[2]}=d\mathcal{A}$ , on the other hand, emerges out of the gravitational action in D+1 dimensions; its coupling parameter corresponds to  $\Delta=4$ .

If one retains in the reduced theory only one of the field strengths  $(\mathcal{F}_{[2]}, G_{[n-1]}, G'_{[n]})$ , together with its corresponding scalar-field combination, then one finds oneself back in the situation described by our general action (2.1), and then the p brane solutions obtained for the general case in Sec. 2 immediately become applicable. Moreover, since retaining only one (field strength, scalar) combination in this way effects a consistent truncation of the theory, solutions to this simple truncated system are also solutions to the untruncated theory, and indeed are also solutions to the original (D+1)-dimensional theory, since the Kaluza-klein dimensional reduction is also a consistent truncation.

The p-brane solutions are ideally structured for Kaluza-Klein reduction, because they are independent of the "worldvolume"  $x^{\mu}$  coordinates. Accordingly, one may let the reduction coordinate z be one of the  $x^{\mu}$ . Consequently,

the only thing that needs to be done to such a solution in order to reinterpret it as a solution of the reduced system (5.7) is to perform a Weyl rescaling on it in order to be in accordance with the form given in the Kaluza-Klein ansatz, which was adjusted so as to maintain the Einstein-frame form of the gravitational term in the action.

After making such a reinterpretation, elementary/solitonic p-branes in (D+1) dimensions give rise to elementary/solitonic (p-1)-branes in D dimensions, corresponding to the same value of  $\Delta$ , as one can see from (5.8,5.9). Note that in this process, the quantity  $\tilde{d}$  is conserved, since both D and d reduce by one. Reinterpretation of p brane solutions in this way, corresponding to standard Kaluza-Klein reduction on a worldvolume coordinate, proceeds diagonally on a D versus d plot, and hence is referred to as diagonal dimensional reduction. This procedure is the analogue, for supergravity field-theory solutions, of the procedure of double dimensional reduction  $^{20}$  for p-brane worldvolume actions, which can be taken to constitute the  $\delta$ -function sources for singular p-brane solutions, coupled in to resolve the singularities.

# 6 Multiple field-strength solutions

Upon Kaluza-Klein dimensional reduction by repeated single steps down to D dimensions, the bosonic sector of maximal supergravity (3.1) reduces to  $^{30}$ 

$$I_{D} = \int d^{D}x \sqrt{-g} \Big[ R - \frac{1}{2} (\partial \vec{\phi})^{2} - \frac{1}{48} e^{\vec{a} \cdot \vec{\phi}} F_{[4]}^{2} - \frac{1}{12} \sum_{i} e^{\vec{a}_{i} \cdot \vec{\phi}} (F_{[3]}^{i})^{2}$$

$$- \frac{1}{4} \sum_{i < j} e^{\vec{a}_{ij} \cdot \vec{\phi}} (F_{[2]}^{ij})^{2} - \frac{1}{4} \sum_{i} e^{\vec{b}_{i} \cdot \phi} (\mathcal{F}_{[2]}^{i})^{2}$$

$$- \frac{1}{2} \sum_{i < j < k} e^{\vec{a}_{ijk} \cdot \vec{\phi}} (F_{[1]}^{ijk})^{2} - \frac{1}{2} \sum_{ij} e^{\vec{b}_{ij} \cdot \phi} (\mathcal{F}_{[1]}^{ij})^{2} \Big] + \mathcal{L}_{FFA} ,$$

$$(6.1)$$

where i, j = 1, ..., 11 - D, and field strengths with multiple i, j indices may be taken to be antisymmetric in those indices since these "internal" indices arise in the stepwise reduction procedure, and two equal index values never occur in a multi-index sum. The "straight-backed" field strengths  $F_{[4]}$ ,  $F_{[3]}^i$ ,  $F_{[2]}^{ij}$  and  $F_{[1]}^{ijk}$  are descendants from  $F_{[4]}$  in D = 11. The "calligraphic" field strengths  $\mathcal{F}_{[2]}^i$  are the field strengths for Kaluza-Klein vectors like  $\mathcal{A}_M$  in (5.1) that emerge from the higher-dimensional metric upon dimensional reduction. Once such a Kaluza-Klein vector has appeared, subsequent dimensional reduction gives rise also to 1-form field strengths  $\mathcal{F}_{[1]}^{ij}$  for zero-form gauge potentials  $\mathcal{A}_{[0]}^{ij}$  as a consequence of the usual one-step reduction (5.3) of a 1-form gauge potential.

The scalar fields  $\vec{\phi}$  that appear in the exponential prefactors in (6.1) form an (11-D)-vector of fields that may be called "dilatonic" scalars. For each field strength occurring in (6.1), there is a corresponding "dilaton vector" of coefficients determining the linear combination of the dilatonic scalars appearing in its exponential prefactor. For the 4-, 3-, 2- and 1-form "straight-backed" field strengths emerging from  $F_{[4]}$  in D=11, these coefficients are denoted correspondingly  $\vec{a}, \vec{a}_i, \vec{a}_{ij}$  and  $\vec{a}_{ijk}$ ; for the "calligraphic" field strengths corresponding to Kaluza-Klein vectors and zero-form gauge potentials emerging out of the metric, these are denoted  $\vec{b}_i$  and  $\vec{b}_{ij}$  correspondingly. Thankfully, not all of these dilaton vectors are independent, and in fact, they may all be expressed in terms of the 4-form and 3-form dilaton vectors  $\vec{a}$  and  $\vec{a}_{ij}$ :<sup>30</sup>

$$\vec{a}_{ij} = \vec{a}_i$$
  $b_i = -\vec{a}_i + \vec{a}$   $\vec{a}_{ijk} = \vec{a}_i + \vec{a}_j + \vec{a}_k - 2\vec{a}$   $\vec{b}_{ij} = -\vec{a}_i + \vec{a}_j$  (6.2)

Another important feature of the dilaton vectors is that they satisfy the following dot product relations:

$$\vec{a} \cdot \vec{a} = \frac{2(11 - D)}{D - 2}$$

$$\vec{a} \cdot \vec{a}_{i} = \frac{2(8 - D)}{D - 2}$$

$$\vec{a}_{i} \cdot \vec{a}_{j} = 2\delta_{ij} + \frac{2(6 - D)}{D - 2}$$
(6.3)

Throughout this review, we have emphasized consistent truncations in making simplifying restrictions of complicated systems of equations, so that the solutions of a simplified system are nonetheless perfectly valid solutions of the more complicated untruncated system. Once again, with the equations of motion following from (6.1) we face a complicated system that calls for analysis in simplified subsectors, so that we may use the solutions already found in our general study of the action (2.1). Accordingly, we now seek a consistent truncation to include just one dilatonic scalar combination  $\phi$  and one rank-n field strength combination  $F_{[n]}$ , constructed out of a certain number N of "retained" field strengths  $F_{\alpha\,[n]}$ ,  $\alpha=1,\ldots,N$ , (possibly a straight-backed/calligraphic mixture) selected from those appearing in (6.1), all the rest being set to zero.<sup>30</sup> Thus, we let

$$\vec{\phi} = \vec{n}\phi + \vec{\phi}_{\perp} \ , \tag{6.4}$$

where  $\vec{n} \cdot \vec{\phi}_{\perp} = 0$ ; in the truncation we seek to set consistently  $\vec{\phi}_{\perp} = 0$ .

Consistency for the field strengths  $F_{\alpha [n]}$  requires them to be proportional.<sup>30</sup> We shall let the dot product matrix for the retained field strengths be denoted

 $M_{\alpha\beta}=:\vec{a}_{\alpha}\cdot\vec{a}_{\beta}$ . Consistency of the truncation requires that the  $\phi_{\perp}$  field equation be satisfied:

$$\Box \vec{\phi}_{\perp} - \sum_{\alpha} \Pi_{\perp} \cdot \vec{a}_{\alpha} (F_{\alpha [n]})^2 = 0 , \qquad (6.5)$$

where  $\Pi_{\perp}$  is the projector into the dilaton-vector subspace orthogonal to the retained dilaton direction  $\vec{n}$ . Setting  $\vec{\phi}_{\perp} = 0$  in (6.5) and letting the retained  $F_{\alpha[n]}$  be proportional, one sees that achieving consistency is hopeless unless all the  $e^{\vec{a}_{\alpha} \cdot \vec{\phi}}$  prefactors are the same, requiring

$$\vec{a}_{\alpha} \cdot \vec{n} = a \quad \forall \alpha = 1, \dots, N ,$$
 (6.6)

where the constant a will play the role of the dilatonic scalar coefficient in the reduced system (2.1). Given a set of dilaton vectors for retained field strengths satisfying (6.6), consistency of (6.5) with setting  $\vec{\phi}_{\perp} = 0$  requires

$$\Pi_{\perp} \cdot \sum_{\alpha} \vec{a}_{\alpha} (F_{\alpha[n]})^2 = 0 . \tag{6.7}$$

This equation requires, for every point  $x^M$  in spacetime, that the combination  $\sum_{\alpha} \vec{a}_{\alpha}(F_{\alpha[n]})^2$  be parallel to  $\vec{n}$  in the dilaton-vector space. Combining this with the requirement (6.6), one has

$$\sum_{\alpha} \vec{a}_{\alpha}(F_{\alpha[n]})^2 = a\vec{n} \sum_{\alpha} (F_{\alpha[n]})^2 . \tag{6.8}$$

Taking then a dot product of this with  $\vec{a}_{\beta}$ , one has

$$\sum_{\alpha} M_{\beta\alpha} (F_{\alpha [n]})^2 = a^2 \sum_{\alpha} (F_{\alpha [n]})^2 . \tag{6.9}$$

Detailed analysis <sup>30</sup> shows it to be sufficient to consider the cases where  $M_{\alpha\beta}$  is invertible, so by applying  $M_{\alpha\beta}^{-1}$  to (6.9), one finds

$$(F_{\alpha[n]})^2 = a^2 \sum_{\beta} M_{\alpha\beta}^{-1} \sum_{\gamma} (F_{\gamma[n]})^2$$
 (6.10)

Summing on  $\alpha$ , one has

$$a^2 = (\sum_{\alpha,\beta} M_{\alpha\beta}^{-1})^{-1} ;$$
 (6.11)

one then defines the retained field-strength combination  $F_{[n]}$  so that

$$(F_{\alpha[n]})^2 = a^2 \sum_{\beta} M_{\alpha\beta}^{-1}(F_{[n]})^2 . \tag{6.12}$$

The only remaining requirement for consistency of the truncation to the simplified  $(g_{MN}, \phi, F_{[n]})$  system (2.1) arises from the necessity to ensure that the variation of the  $\mathcal{L}_{FFA}$  term in (6.1) is not inconsistent with setting to zero the discarded dilatonic scalars and gauge potentials. In general, this imposes a somewhat complicated requirement. For the purposes of the present review, however, we shall concentrate on either purely-electric solutions satisfying the elementary ansatz (2.4) or purely-magnetic solutions satisfying the solitonic ansatz (2.6). As one can see by inspection, for pure electric or magnetic solutions of these sorts, the terms that are dangerous for consistency arising from the variation of  $\mathcal{L}_{FFA}$  all vanish. Thus, for such solutions one may safely ignore the complications of the  $\mathcal{L}_{FFA}$  term. This restriction to pure electric or magnetic solutions does, however, leave out the very interesting cases of dyonic solutions that exist in D=8 and D=4, upon which we shall comment briefly later on.

After truncating to the system (2.1), the analysis proceeds as in Section 2. It turns out <sup>30</sup> that *supersymmetric* p-brane solutions arise when the matrix  $M_{\alpha\beta}$  for the retained  $F_{\alpha[n]}$  satisfies

$$M_{\alpha\beta} = 4\delta_{\alpha\beta} - \frac{2d\tilde{d}}{D-2} , \qquad (6.13)$$

and the corresponding  $\Delta$  value for  $F_{[n]}$  is

$$\Delta = \frac{4}{N} \,\,, \tag{6.14}$$

where we recall that N is the number of retained field strengths. A generalization of this analysis leads to a classification of solutions with more than one independent retained scalar-field combination  $^{30}$ , but to pursue that would go beyond the focus of the present review, where we shall limit ourselves to the "single-scalar" context.

#### 7 Multi-center solutions and vertical dimensional reduction

As we saw above in Section 5, the translation Killing symmetries of p-brane solutions allow a simultaneous interpretation of such solutions as belonging to several different supergravity theories, related one to another by Kaluza-Klein dimensional reduction. For the original single p-brane solutions (2.23), the only available translational Killing symmetries are those in the worldvolume directions, which we exploited in describing diagonal dimensional reduction. One may, however, generalize the basic solutions (2.23) by replacing the harmonic

function H(y) in (2.23c) by a different solution of the Laplace equation (2.19). Thus, one can easily extend the family of p-brane solutions to multi-center p-brane solutions by taking the harmonic function to be

$$H(y) = 1 + \sum_{\alpha} \frac{k_{\alpha}}{|\vec{y} - \vec{y}_{\alpha}|^{\tilde{d}}} \qquad k_{\alpha} > 0 .$$
 (7.1)

Once again, the integration constant has been adjusted to make  $H_{|_{\infty}}=1 \leftrightarrow \phi_{|_{\infty}}=0$ . The generalized solution (7.1) corresponds to parallel and similarly-oriented p-branes, with all charges  $\lambda_{\alpha}=2\tilde{d}k_{\alpha}/\sqrt{\Delta}$  required to be positive in order to avoid naked singularities. The "centers" of the individual "leaves" of this solution are at the points  $y=y_{\alpha}$ , where  $\alpha$  ranges over any number of centers. The metric and the electric-case antisymmetric tensor gauge potential corresponding to (7.1) are given again in terms of H(y) by (2.23a,2.24). In the solitonic case, the ansatz (2.6) needs to be modified so as to accommodate the multi-center form of the solution:

$$F_{m_1...m_n} = -\tilde{d}^{-1} \epsilon_{m_1...m_n p} \partial_p \sum_{\alpha} \frac{\lambda_{\alpha}}{|\vec{y} - \vec{y}_{\alpha}|^{\tilde{d}}} , \qquad (7.2)$$

which ensures validity of the Bianchi identity just as well as (2.6) does. The mass/(unit p-volume) density is now

$$\mathcal{E} = \frac{1}{2\sqrt{\Delta}} \sum \alpha \lambda_{\alpha} , \qquad (7.3)$$

while the total electric or magnetic charge is given by  $\frac{1}{4} \sum \lambda_{\alpha}$ , so the Bogomol'ny bounds (4.14) are saturated just as they are for the single-center solutions (2.23). Since the multi-center solutions given by (7.1) satisfy the same supersymmetry-preservation conditions on the metric and antisymmetric tensor as (2.23), the multi-center solutions leave the same amount of supersymmetry unbroken as the single-center solution.

From a mathematical point of view, the multi-center solutions (7.1) exist owing to the properties of the Laplace equation (2.19). From a physical point of view, however, these static solutions exist as a result of *cancellation* between *attractive* gravitational and scalar-field forces against *repulsive* antisymmetric-tensor forces for the similarly-oriented p-brane "leaves."

The multi-center solutions given by (7.1) can now be used to prepare solutions adapted to dimensional reduction in the *transverse* directions. This combination of modifying the solution followed by dimensional reduction on a transverse coordinate is called *vertical* dimensional reduction <sup>5</sup> because it

relates solutions vertically on a D versus d plot. In order to do this, we need first to develop a translation invariance in the transverse reduction coordinate. This can be done by "stacking" up identical p branes using (7.1) in a periodic array, i.e. by letting the integration constants  $k_{\alpha}$  all be equal, and aligning the "centers"  $y_{\alpha}$  along some axis, e.g. the  $\hat{z}$  axis. Singling out one "stacking axis" in this way clearly destroys the overall isotropic symmetry of the solution, but, provided the centers are all in a line, the solution will nonetheless remain isotropic in the D-d-1 dimensions orthogonal to the stacking axis. Taking the limit of a densely-packed infinite stack of this sort, one has

$$\sum_{\alpha} \frac{k_{\alpha}}{|\vec{y} - \vec{y}_{\alpha}|^{\tilde{d}}} \longrightarrow \int_{-\infty}^{+\infty} \frac{kdz}{(\hat{r}^2 + z^2)^{\tilde{d}/2}} = \frac{\tilde{k}}{\tilde{r}^{\tilde{d}-1}}$$
 (7.4a)

$$\hat{r}^2 = \sum_{m=d}^{D-2} y^m y^m$$

$$\hat{k} = \frac{\sqrt{\pi} k \Gamma(\tilde{d} - \frac{1}{2})}{2\Gamma(\tilde{d})},$$
(7.4b)

$$\hat{k} = \frac{\sqrt{\pi k \Gamma(\tilde{d} - \frac{1}{2})}}{2\Gamma(\tilde{d})}, \qquad (7.4c)$$

where  $\hat{r}$  in (7.4b) is the radial coordinate for the D-d-1 residual isotropic transverse coordinates. After a conformal rescaling to maintain the Einstein frame for the solution, one can finally reduce on the coordinate z along the stacking axis.

After stacking and reduction in this way, one obtains a p-brane solution with the same worldvolume dimension as the original higher-dimensional solution that was stacked up. Since the same antisymmetric tensors are used here to support both the stacked and the unstacked solutions, and since  $\Delta$ is preserved under dimensional reduction, it follows that vertical dimensional reduction from D to D-1 spacetime dimensions preserves the value of  $\Delta$  just like the diagonal reduction discussed in the previous section. Note that, under vertical reduction, the worldvolume dimension d is preserved, but  $\tilde{d} = D - d - 2$ is reduced by one with each reduction step.

f Similar procedures have been considered in a number of articles in the literature; see, e.q. Refs.<sup>31</sup>

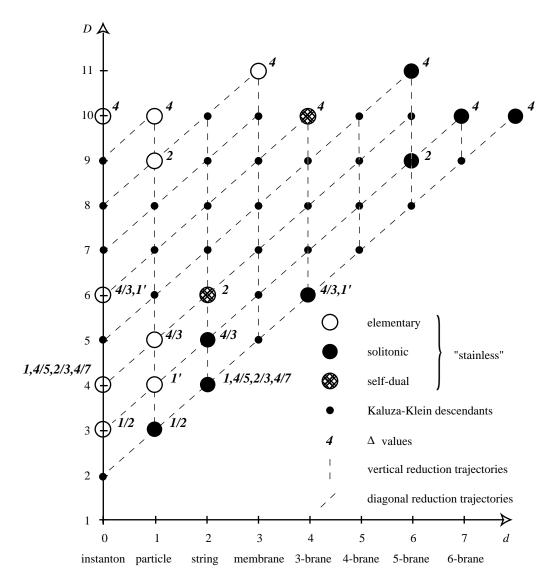


Figure 4: Brane-scan of supergravity p-brane solutions  $(p \leq (D-3))$ 

Combining the diagonal and vertical dimensional reduction trajectories of "descendant" solutions, one finds the general picture given in the plot of Figure 4. In this plot of spacetime dimension D versus worldvolume dimension d, reduction families emerge from certain basic solutions that cannot be "oxidized" to higher-dimensional solutions, and hence can be called "stainless" p-branes. In Figure 4, these solutions are indicated by the large circles, with the corresponding  $\Delta$  values shown adjacent. The indication of the elementary or solitonic type of solution relates to solutions of supergravity theories in versions with the lowest possible choice of rank  $(n \leq D/2)$  for the supporting field strength, obtainable by appropriate dualization.

# 8 Beyond the (D-3)-brane barrier: Scherk-Schwarz reduction and domain walls

The process of vertical dimensional reduction described in the previous section proceeds uneventfully until one makes the reduction from a (D, d = D - 3) solution to a (D-1, d = D-3) solution. In this step, the integral (7.4) contains an additive divergence and needs to be renormalized. This is easily handled by putting finite limits  $\pm L$  on the integral, which becomes  $\int_{-L}^{L} d\tilde{z} (r^2 + \tilde{z}^2)^{-1/2}$ , and then by subtracting a divergent term  $2 \ln L$  before taking the limit  $L \to \infty$ . Then the integral gives the expected  $\ln \hat{r}$  harmonic function appropriate to two transverse dimensions.

Continuing on down, one may similarly make one more step of reduction, from the (D-1,d=D-3) solution obtained above, in an attempt to create a solution with dimensions (D-2,d=D-3), i.e. a (D-2)-brane, or domain wall. In the process of vertical dimensional reduction, this step again gives rise to an additive divergence: the integral  $\int_{-L}^{L} dz \ln(y^2+z^2)$  needs to be renormalized by subtracting a divergent term  $4L(\ln L-1)$ . After subsequently performing the integral, the resulting harmonic H(y) becomes linear in the one remaining transverse coordinate.

While the above mathematical procedure of vertical dimensional reduction to produce a (D-2)-brane proceeds apparently without serious complication, analyzing the physics of the situation needs some care.<sup>6</sup> There are three things about the reduction from a (D-1, d=D-3) solution (a (D-3)-brane) to a (D-2, d=D-3) solution (a (D-2)-brane) that require special attention. First, let us note that both the (D-3) brane and its descendant (D-2)-brane have harmonic functions H(y) that blow up at infinity. For the (D-3)-brane,

 $<sup>^</sup>g$ Such solutions, with worldvolume dimension two less than the spacetime dimension, will be referred to generally as (D-3)-branes, irrespective of whether the spacetime dimension is D or not.

however this is not in itself particularly remarkable, because, as one can see by inspection of (2.23) for this case, the metric asymptotically tends to a locally flat space as  $r \to \infty$ , and also in this limit the dilatonic scalar  $\phi$  and antisymmetric-tensor one-form field strength

$$F_m = -\epsilon_{mn}\partial_n H \tag{8.1}$$

tend asymptotically to zero. The expression (8.1) for the field strength, however, shows that the next reduction step to the (D-2, d=D-3) solution has a significant new feature: in stacking up (D-3)-branes prior to the vertical reduction, producing a linear harmonic function in the transverse coordinate y,

$$H(y) = \text{const.} + my , \qquad (8.2)$$

the field strength (8.1) has a *constant* component along the stacking axis  $\leftrightarrow$  reduction direction z,

$$F_z = -\epsilon_{zy}\partial_y H = m , \qquad (8.3)$$

that implies an unavoidable dependence  $^h$  of the corresponding zero-form gauge potential on the reduction coordinate:

$$A_{[0]}(x, y, z) = mz + \chi(x, y) . \tag{8.4}$$

From a Kaluza-Klein point of view, the unavoidable linear dependence of a gauge potential on the reduction coordinate given in (8.4) appears to be problematic. Throughout this review, we have dealt only with consistent) Kaluza-Klein reductions, for which solutions of the reduced theory are also solutions of the unreduced theory. Generally, retaining any dependence on a reduction coordinate will lead to an  $inconsistent\ truncation$  of the theory: attempting to impose a z dependence of the form given in (8.4) prior to varying the Lagrangian will give a result different from that of imposing this dependence in the field equations after variation.

Before showing how this consistency problem can sometimes be avoided, let us consider two other facets of the problem with the vertical reduction of (D-3)-branes. Firstly, the asymptotic metric of a (D-3)-brane is not a globally flat space, but only a locally flat space. This distinction means that there is in general a deficit solid angle at transverse infinity, which is related to the total mass density of the (D-3)-brane.<sup>32</sup> This means that any

 $<sup>^</sup>h$ Note that this vertical reduction from a (D-3)-brane to a (D-2)-brane is the first time one is forced to accept a dependence on the reduction coordinate z; in all higher-dimensional vertical reductions, z dependence can be removed by a gauge transformation. The zero-form gauge potential in (8.4) does not have such a gauge symmetry, however.

attempt to stack up (D-3)-branes within a standard supergravity theory will soon consume the entire solid angle at transverse infinity, thus destroying the asymptotic spacetime by such a construction.

The second facet of the problem with (D-2)-branes in ordinary supergravity theories is simply stated: starting from the p-brane ansatz (2.3,2.6) and searching for (D-2) branes in ordinary massless supergravity theories, one simply doesn't find any such solutions.

The resolution of all these difficulties happens together, for in blindly performing a Kaluza-Klein reduction with an ansatz like (8.4), one is in fact making a departure from the set of standard massless supergravity theories. In order to understand this, let us concentrate on the problem of consistency of the Kaluza-Klein reduction. As we have seen, consistency of any restriction on the field variables with respect to the equations of motion means that the restriction may either be imposed on the field variables in the action prior to variation to derive the equations of motion, or may be imposed on the field variables in the equations of motion after variation, with an equal effect. In that case, solutions obeying the restriction will also be solutions of the general unrestricted equations of motion.

The most usual guarantee of consistency in Kaluza-Klein dimensional reduction is achieved by restricting the field variables to carry zero charge with respect to some conserved current, e.g. momentum in the reduction dimension. But this is not the only way in which consistency may be achieved. In the present case, retaining a linear dependence on the reduction coordinate as in (8.4) clearly would produce an inconsistent truncation if the reduction coordinate were to appear explicitly in any of the field equations. But this does not imply that a truncation retaining some dependence on the reduction coordinate is necessarily inconsistent just because a gauge potential contains a term linear in that coordinate. Inconsistency of a Kaluza-Klein truncation occurs when the original unrestricted field equations imply a condition that is inconsistent with the reduction ansatz. If a particular gauge potential appears in the action only through its derivative, i.e. through its field strength, then a consistent truncation may also be achieved provided that the restriction on the gauge field implies that the field strength is independent of the reduction coordinate. A zero-form gauge potential on which such a reduction may be carried out, occurring in the action only through its derivative, will be referred to as an axion.

Requiring axionic field strengths to be independent of the reduction coordinate amounts to extending the Kaluza-Klein reduction framework to allow linear dependence of an axionic zero-form potential on the reduction coordinate precisely of the form occurring in (8.4). So, provided  $A_{[0]}$  is an axion,

the reduction (8.4) turns out to be consistent after all. This extension of the Kaluza-Klein ansatz is in fact an instance of Scherk-Schwarz reduction.<sup>33, 34</sup> The basic idea of Scherk-Schwarz reduction is to use an Abelian rigid symmetry of a system of equations, but then to generalize the reduction ansatz by allowing a linear dependence on the reduction coordinate in the parameter of this Abelian symmetry. Consistency is guaranteed by cancellations orchestrated by this Abelian symmetry in field-equation terms where the parameter does not get differentiated. When it does get differentiated, it contributes only a term that is itself independent of the reduction coordinate. In the present case, the Abelian symmetry guaranteeing consistency of (8.4) is a simple shift symmetry  $A_{[0]} \rightarrow A_{[0]} + \text{const.}$ 

Unlike the original implementation of the Scherk-Schwarz reduction idea,  $^{33}$  which used an Abelian U(1) phase symmetry acting on spinors, the Abelian shift symmetry used here commutes with supersymmetry, and hence the reduction does not spontaneously break supersymmetry. Instead, gauge symmetries for some of the antisymmetric tensors will be broken, with a corresponding appearance of mass terms. As with all examples of vertical dimensional reduction, the  $\Delta$  value corresponding to a given field strength is also preserved. Thus, p-brane solutions related by vertical dimensional reduction, even in the enlarged Scherk-Schwarz sense, preserve the same amount of unbroken supersymmetry and have the same value of  $\Delta$ .

It may be necessary to make several redefinitions and integrations by parts in order to reveal the axionic property of a given zero-form, and thus to prepare the theory for a reduction of the form (8.4). This is most easily explained by an example, so let us consider the first possible Scherk-Schwarz reduction in the sequence of theories descending from (3.1), starting in D=9 where the first axion field appears. The Lagrangian for massless D=9 maximal supergravity is obtained by specializing the general dimensionally-reduced action (6.1) given in Section 2 to this case:

$$\mathcal{L}_{9} = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi_{1})^{2} - \frac{1}{2} (\partial \phi_{2})^{2} - \frac{1}{2} e^{\phi_{1} + \frac{3}{\sqrt{7}} \phi_{2}} (\partial \chi)^{2} - \frac{1}{48} e^{\vec{a} \cdot \vec{\phi}} (F_{[4]})^{2} \right. \\
\left. - \frac{1}{2} e^{\vec{a}_{1} \cdot \vec{\phi}} (F_{[3]}^{(1)})^{2} - \frac{1}{2} e^{\vec{a}_{2} \cdot \vec{\phi}} (F_{[3]}^{(2)})^{2} - \frac{1}{4} e^{\vec{a}_{12} \cdot \vec{\phi}} (F_{[2]}^{(12)})^{2} - \frac{1}{4} e^{\vec{b}_{1} \cdot \vec{\phi}} (\mathcal{F}_{[2]}^{(1)})^{2} \\
\left. - \frac{1}{4} e^{\vec{b}_{2} \cdot \vec{\phi}} (\mathcal{F}_{[2]}^{(2)})^{2} \right] - \frac{1}{2} \tilde{F}_{[4]} \wedge \tilde{F}_{[4]} \wedge A_{[1]}^{(12)} - \tilde{F}_{[3]}^{(1)} \wedge \tilde{F}_{[3]}^{(2)} \wedge A_{[3]} , \quad (8.5)$$

where 
$$\chi = \mathcal{A}_{[0]}^{(12)}$$
 and  $\vec{\phi} = (\phi_1, \phi_2)$ .

<sup>&</sup>lt;sup>i</sup>A higher-dimensional Scherk-Schwarz reduction is possible <sup>34</sup> starting from type IIB supergravity in D=10, using the axion appearing in the  $SL(2,\mathbb{R})/SO(2)$  scalar sector of that theory.

Within the scalar sector  $(\vec{\phi}, \chi)$  of (8.5), the dilaton coupling has been made explicit; in the rest of the Lagrangian, the dilaton vectors have the general structure given in (6.2,6.4). The scalar sector of (8.5) forms a nonlinear  $\sigma$ model for the manifold  $GL(2, \mathbb{R})/SO(2)$ . This already makes it appear that one may identify  $\chi$  as an axion available for Scherk-Schwarz reduction. However, account must still be taken of the Chern-Simons structure lurking inside the field strengths in (8.5). In detail, the field strengths are given by

$$F_{[4]} = \tilde{F}_{[4]} - \tilde{F}_{[3]}^{(1)} \wedge \mathcal{A}_{[1]}^{(1)} - \tilde{F}_{[3]}^{(2)} \wedge \mathcal{A}_{[1]}^{(2)} + \chi \tilde{F}_{[3]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)} - \tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)}$$

$$(8.6a)$$

$$F_{[3]}^{(1)} = \tilde{F}_{[3]}^{(1)} - \tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(2)}$$
 (8.6b)

$$F_{[3]}^{(2)} = \tilde{F}_{[3]}^{(2)} + F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} - \chi \tilde{F}_{[3]}^{(1)}$$
 (8.6c)

$$F_{[2]}^{(12)} = \tilde{F}_{[2]}^{(12)} \qquad \mathcal{F}_{[2]}^{(1)} = \mathcal{F}_{[2]}^{(1)} - d\chi \wedge \mathcal{A}_{[1]}^{(2)}$$
 (8.6d)

$$F_{[3]}^{(1)} = \tilde{F}_{[3]}^{(1)} - \tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(2)}$$

$$F_{[3]}^{(2)} = \tilde{F}_{[3]}^{(2)} + F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} - \chi \tilde{F}_{[3]}^{(1)}$$

$$(8.6b)$$

$$F_{[3]}^{(2)} = \tilde{F}_{[3]}^{(2)} + F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} - \chi \tilde{F}_{[3]}^{(1)}$$

$$F_{[2]}^{(12)} = \tilde{F}_{[2]}^{(12)} \qquad \mathcal{F}_{[2]}^{(1)} = \mathcal{F}_{[2]}^{(1)} - d\chi \wedge \mathcal{A}_{[1]}^{(2)}$$

$$\mathcal{F}_{[2]}^{(2)} = \tilde{\mathcal{F}}_{[2]}^{(2)} \qquad \mathcal{F}_{[1]}^{(12)} = d\chi ,$$

$$(8.6e)$$

where the field strengths carrying tildes are the naïve expressions without Chern-Simons corrections, i.e.  $\hat{F}_{n} = dA_{[n-1]}$ . Now the appearance of undifferentiated  $\chi$  factors in (8.6a,c) makes it appear that Scherk-Schwarz reduction would be inconsistent. However, one may eliminate these undifferentiated factors by making the field redefinition

$$A_{[2]}^{(2)} \longrightarrow A_{[2]}^{(2)} + \chi A_{[2]}^{(1)} ,$$
 (8.7)

after which the field strengths (8.6a,c) become

$$F_{[4]} = \tilde{F}_{[4]} - \tilde{F}_{[3]}^{(1)} \wedge \mathcal{A}_{[1]}^{(1)} - \tilde{F}_{[3]}^{(2)} \wedge \mathcal{A}_{[1]}^{(2)} \\ -d\chi \wedge A_{[2]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)} - \tilde{F}_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)}$$
(8.8a)

$$F_{[3]}^{(2)} = \tilde{F}_{[3]}^{(2)} + F_{[2]}^{(12)} \wedge \mathcal{A}_{[1]}^{(1)} + d\chi \wedge A_{[2]}^{(1)}, \qquad (8.8c)$$

the rest of (8.6) remaining unchanged.

After making the field redefinitions (8.7), the axion field  $\chi = A_{[0]}^{(12)}$  is ready for application of the Scherk-Schwarz reduction ansatz (8.4). The coefficient of the term linear in the reduction coordinate z has been denoted m because it carries the dimensions of mass, and correspondingly its effect on the reduced action is to cause the appearance of mass terms. Applying (8.4) to the D=9 Lagrangian, one obtains the D=8 reduced Lagrangian<sup>6</sup>

$$\mathcal{L}_{8 \text{ ss}} = \sqrt{-g} \Big[ R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{1}{2} (\partial \phi_3)^2 - \frac{1}{2} e^{\vec{b}_{12} \cdot \vec{\phi}} (\partial \chi - m \mathcal{A}_{[1]}^{(3)})^2 \\
- \frac{1}{2} e^{\vec{b}_{13} \cdot \vec{\phi}} (\partial \mathcal{A}_{[0]}^{(13)} + m \mathcal{A}_{[1]}^{(2)})^2 - \frac{1}{2} e^{\vec{b}_{23} \cdot \vec{\phi}} (\partial \mathcal{A}_{[0]}^{(23)})^2 - \frac{1}{2} e^{\vec{a}_{123} \cdot \vec{\phi}} (\partial \mathcal{A}_{[0]}^{(123)})^2 \\
- \frac{1}{48} e^{\vec{a} \cdot \vec{\phi}} (F_{[4]} - m \mathcal{A}_{[2]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)} \wedge \mathcal{A}_{[1]}^{(3)})^2 - \frac{1}{12} e^{\vec{a}_1 \cdot \vec{\phi}} (F_{[3]}^{(1)})^2 \\
- \frac{1}{12} e^{\vec{a}_2 \cdot \vec{\phi}} (F_{[3]}^{(2)} + m \mathcal{A}_{[2]}^{(1)} \wedge \mathcal{A}_{[1]}^{(3)})^2 - \frac{1}{12} e^{\vec{a}_3 \cdot \vec{\phi}} (F_{[3]}^{(3)} + m \mathcal{A}_{[2]}^{(1)} \wedge \mathcal{A}_{[1]}^{(2)})^2 \\
- \frac{1}{4} e^{\vec{a}_{12} \cdot \vec{\phi}} (F_{[2]}^{(12)})^2 - \frac{1}{4} e^{\vec{a}_{13} \cdot \vec{\phi}} (F_{[2]}^{(13)})^2 - \frac{1}{4} e^{\vec{b}_{23} \cdot \vec{\phi}} (F_{[2]}^{(23)} + m \mathcal{A}_{[2]}^{(1)})^2 \\
- \frac{1}{4} e^{\vec{b}_1 \cdot \vec{\phi}} (\mathcal{F}_{[2]}^{(1)} - m \mathcal{A}_{[1]}^{(2)} \wedge \mathcal{A}_{[1]}^{(3)})^2 - \frac{1}{4} e^{\vec{b}_2 \cdot \vec{\phi}} (\mathcal{F}_{[2]}^{(2)})^2 - \frac{1}{4} e^{\vec{b}_3 \cdot \vec{\phi}} (\mathcal{F}_{[2]}^{(3)})^2 \\
- \frac{1}{2} m^2 e^{\vec{b}_{123} \cdot \vec{\phi}} \Big] + \mathcal{L}_{FFA} , \tag{8.9}$$

where the dilaton vectors are now those appropriate for D=8. It is apparent from (8.9) that the fields  $\mathcal{A}^{(3)}_{[1]}$ ,  $\mathcal{A}^{(2)}_{[1]}$  and  $A^{(1)}_{[2]}$  have now become massive, absorbing in the process  $\chi$ ,  $\mathcal{A}^{(13)}_{[0]}$  and  $A^{(23)}_{[1]}$ . Specifically, these fields are absorbed by making the following gauge transformations:

$$\mathcal{A}_{[1]}^{(3)} \longrightarrow \mathcal{A}_{[1]}^{(3)} + \frac{1}{m} d\chi 
\mathcal{A}_{[1]}^{(2)} \longrightarrow \mathcal{A}_{[1]}^{(2)} - \frac{1}{m} d\mathcal{A}_{[0]}^{(13)} 
\mathcal{A}_{[2]}^{(1)} \longrightarrow \mathcal{A}_{[2]}^{(1)} - \frac{1}{m} d\mathcal{A}_{[1]}^{(23)} .$$
(8.10)

Note that  $\mathcal{A}^{(3)}_{[1]}$  is the Kaluza-Klein vector field corresponding to this  $D=9 \to D=8$  reduction, and that in becoming massive it consumes the axion  $\chi$  on which the Scherk-Schwarz reduction was performed; this is a general feature of such reductions. Furthermore, the appearance of the derivative of the reduction axion in Chern-Simons corrections to other field strengths gives rise to further spontaneous breakings, in this case giving masses to  $\mathcal{A}^{(2)}_{[1]}$  and  $\mathcal{A}^{(1)}_{[2]}$ .

As one descends through the available spacetime dimensions for supergravity theories, the number of axionic scalars available for a Scherk-Schwarz reduction step increases. The numbers of axions are given in the following table:

Table 1: Supergravity axions versus spacetime dimension.

D	9	8	7	6	5	4
$N_{\rm axions}$	1	4	10	20	36	63

Each of these axions gives rise to a distinct massive supergravity theory upon Scherk-Schwarz reduction, and each of these reduced theories has its own pattern of mass generation. In addition, once a Scherk-Schwarz reduction step has been performed, the resulting theory can be further reduced using ordinary Kaluza-Klein reduction. Moreover, the Scherk-Schwarz and ordinary Kaluza-Klein processes do not commute, so the number of theories obtained by the various combinations of Scherk-Schwarz and ordinary dimensional reduction is cumulative. In addition, there are numerous possibilities of performing Scherk-Schwarz reduction simultaneously on a number of axions. As one can see from the D = 9 example, field redefinitions like (8.7) need to be made in order to cover axions with derivatives prior to Scherk-Schwarz reduction, and these redefinitions need to be consistent if one is to perform such reduction on multiple axions. The set of axions that can be simultaneously covered by derivatives always includes the full set of Ramond-Ramond (R-R) sector axions, i.e. those of the forms  $\mathcal{A}_{[0]}^{(1a)}$  or  $A_{[0]}^{(abc)}$ ,  $a,b=2,\ldots,11-D$ , which are  $2^{9-D}$  in number. There are additional possibilities involving NS-NS sector axions; for example, in D=8, three out of the four axions shown in Table 1 may be simultaneously covered by derivatives: the two R-R axions plus one NS-NS axion. For further details on this panoply of Scherk-Schwarz reduction possibilities, we refer the reader to Ref.<sup>6</sup>

For our present purposes, the important feature of theories obtained by Scherk-Schwarz reduction is the appearance of cosmological potential terms such as the penultimate term in Eq. (8.9). Such terms may be considered within the context of our simplified action (2.1) by letting the rank n of the field strength take the value zero. Accordingly, by consistent truncation of (8.9) or of one of the many theories obtained by Scherk-Schwarz reduction in lower dimensions, one may arrive at the simple Lagrangian

$$\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2} m^2 e^{a\phi} \right] . \tag{8.11}$$

Since the rank of the form here is n=0, the elementary/electric type of solution would have worldvolume dimension d=-1, which is not very sensible, but the solitonic/magnetic solution has  $\tilde{d}=D-1$ , corresponding to a p=D-2 brane, or domain wall, as expected. Relating the parameter a in (8.11) to the

reduction-invariant parameter  $\Delta$  by the standard formula (2.17) gives  $\Delta$  $a^2 - 2(D-1)/(D-2)$ ; taking the corresponding p = D-2 brane solution from (2.23), one finds<sup>j</sup>

$$ds^{2} = H^{\frac{4}{\Delta(D-2)}} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + H^{\frac{4(D-1)}{\Delta(D-2)}} dy^{2}$$
(8.12a)  

$$e^{\phi} = H^{2a/\Delta},$$
(8.12b)

$$e^{\phi} = H^{2a/\Delta} , \qquad (8.12b)$$

where the harmonic function H(y) is now a linear function of the single transverse coordinate, in accordance with (8.2). The curvature of the metric (8.12a)tends to zero at large values of |y|, but it diverges if H tends to zero. This latter singularity can be avoided by taking H to be

$$H = \text{const.} + M|y| \tag{8.13}$$

where  $M = \frac{1}{2}m\sqrt{\Delta}$ . With the choice (8.13), there is just a delta-function singularity at the location of the domain wall at y=0, corresponding to the discontinuity in the gradient of H.

The domain-wall solution (8.12,8.13) has the peculiarity of tending asymptotically to flat space as  $|y| \to \infty$ , within a theory that does not itself admit flat space as a vacuum solution. In fact, the theory (8.11) does not even admit a maximally-symmetric solution, owing to the complication of the cosmological potential. The domain-wall solution (8.12,8.13), however, manages to "cancel" this potential at transverse infinity, allowing at least asymptotic flatness for this solution. This brings us back to the other facets of the consistency problem for vertical dimensional reduction to produce (D-2)-branes as discussed at the beginning of this section. There is no inconsistency between the existence of domain-wall solutions like (8.12,8.13) and the inability to find such solutions in standard supergravity theories, or with the conical spacetime character of (D-3)-branes because the domain walls exist in a quite different context of massive supergravity theories like (8.9) with a vacuum structure different from that of standard massless supergravities.

#### Duality symmetries and the classification of p-branes

#### Supergravity duality symmetries

As one can see from our discussion of Kaluza-Klein dimensional reduction in Section 5, progression down to lower dimensions D causes the number of dilatonic scalars  $\phi$  and also the number of zero-form potentials of 1-form field

<sup>&</sup>lt;sup>j</sup>Domain walls solutions such as (8.12) in supergravity theories were found for the D=4case in Ref.<sup>35</sup> and a recent review of them has been given in Ref.<sup>36</sup>

strengths to proliferate. When one reaches D=4, for example, a total of 70 such spin-zero fields has accumulated. In D=4, the maximal (N=8) supergravity equations of motion have a linearly-realized  $H=\mathrm{SU}(8)$  symmetry; this is also the automorphism symmetry of the D=4, N=8 supersymmetry algebra relevant to the (self-conjugate) supergravity multiplet. In formulating this symmetry, it is necessary to consider complex self-dual and anti-self-dual combinations of the 2-form field strengths, which are the highest-rank field strengths in D=4, higher ranks having been eliminated by the reduction or by dualization. Using two-component notation for the D=4 spinors, these combinations transform as  $F_{\alpha\beta}^{[ij]}$  and  $\bar{F}_{\dot{\alpha}\dot{\beta}\,[ij]}, i, j=1,\ldots,8, i.e.$  as a complex 28-dimensional dimensional representation of  $\mathrm{SU}(8)$ . Since this complex representation can be carried only by the complex field-strength combinations and not by the 1-form gauge potentials, it cannot be locally formulated at the level of the gauge potentials or of the action, where only an  $\mathrm{SO}(8)$  symmetry is apparent.

Taking all the spin-zero fields together, one finds that they form a rather impressive nonlinear  $\sigma$ -model on a 70-dimensional manifold. Anticipating that this manifold must be a coset space with  $H=\mathrm{SU}(8)$  as the linearly-realized denominator group, Cremmer and Julia <sup>37</sup> deduced that it had to be the manifold  $\mathrm{E}_{7(+7)}/\mathrm{SU}(8)$ ; since the dimension of  $\mathrm{E}_7$  is 133 and that of  $\mathrm{SU}(8)$  is 63, this gives a 70-dimensional manifold. Correspondingly, a nonlinearly-realized  $\mathrm{E}_{7(+7)}$  symmetry also appears as an invariance of the  $D=4,\,N=8$  maximal supergravity equations of motion. Such nonlinearly-realized symmetries of supergravity theories have always had a somewhat mysterious character. They arise in part out of general covariance in the higher dimensions, from which supergravities arise by dimensional reduction, but this is not enough: such symmetries act transitively on the  $\sigma$ -model manifolds, mixing fields arising both from the metric and from the reduction of the D=11 3-form potential  $A_{[3]}$  in (3.1).

In dimensions  $4 \leq D \leq 9$ , maximal supergravity has the sets of  $\sigma$ -model nonlinear G and linear H symmetries shown in Table 2. In all cases, the spinzero fields take their values in "target" manifolds G/H. Just as the asymptotic value at infinity of the metric defines the reference, or "vacuum" spacetime with respect to which integrated charges and energy/momentum are defined, so do the asymptotic values of the spin-zero fields define the "scalar vacuum." These asymptotic values are referred to as the moduli of the solution. In string theory, these moduli acquire interpretations as the  $coupling\ constants$  and vacuum  $\theta$ -angles of the theory. Once these are determined for a given "vacuum," the classification symmetry that organizes the distinct solutions of the theory into families with the same energy must be a subgroup of the  $little\ group$ , or isotropy

group, of the vacuum. In ordinary General Relativity with asymptotically flat spacetimes, the analogous group is the spacetime Poincaré group times the appropriate classifying symmetry for internal symmetries, such as the group of rigid (*i.e.* constant-parameter) Yang-Mills gauge transformations.

The isotropy group of any point on a coset manifold G/H is just H, so this is the classical "internal" classifying symmetry for supergravity.

D	G	H
9	$\mathrm{GL}(2, {\rm I\!R})$	SO(2)
8	$SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$	$SO(3) \times SO(2)$
7	$\mathrm{SL}(5,{\rm I\!R})$	SO(5)
6	SO(5,5)	$SO(5) \times SO(5)$
5	$E_{6(+6)}$	USP(8)
4	$E_{7(+7)}$	SU(8)

Table 2: Supergravity  $\sigma$ -model symmetries.

# 9.2 An example of duality symmetry: D = 8 supergravity

In maximal D=8 supergravity, one sees from Table 2 that  $G=\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$  and the isotropy group is  $H=\mathrm{SO}(3)\times\mathrm{SO}(2)$ . We have an (11-3=8) vector of dilatonic scalars as well as a singlet  $F_{[1]}^{ijk}$  and a triplet  $\mathcal{F}_{[1]}^{ij}$  (i,j,k=1,2,3) of 1-form field strengths for zero-form potentials. Taken all together, we have a manifold of dimension 7, which fits in precisely with the dimension of the  $(\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/(\mathrm{SO}(3)\times\mathrm{SO}(2))$  coset-space manifold: 8+3-(3+1)=7.

Owing to the direct-product structure, we may for the time being eliminate the 5-dimensional  $SL(3,\mathbb{R})/SO(3)$  sector and consider for simplicity just the 2-dimensional  $SL(2,\mathbb{R})/SO(2)$  sector. Here is the relevant part of the action:<sup>38</sup>

$$I_8^{\text{SL}(2)} = \int d^8x \sqrt{-g} \Big[ R - \frac{1}{2} \nabla_{\scriptscriptstyle M} \sigma \nabla^{\scriptscriptstyle M} \sigma - \frac{1}{2} e^{-2\sigma} \nabla_{\scriptscriptstyle M} \chi \nabla^{\scriptscriptstyle M} \chi - \frac{1}{2 \cdot 4!} e^{\sigma} (F_{[4]})^2 - \frac{1}{2 \cdot 4!} \chi F_{[4]}^* F_{[4]} \Big]$$
(9.1)

where  ${}^*F^{MNPQ}=1/(4!\sqrt{-g})\epsilon^{MNPQx_1x_2x_3x_4}F_{x_1x_2x_3x_4}$  (the  $\epsilon^{[8]}$  is a density, so purely numerical).

On the scalar fields  $(\sigma, \chi)$ , the SL(2,  $\rm I\!R$ ) symmetry acts as follows: let  $\lambda = \chi + i e^{\sigma}$ ; then

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{9.2}$$

with ab-cd=1 is an element of  $\mathrm{SL}(2,\mathbbm{R})$  and acts on  $\lambda$  by the fractional-linear transformation

$$\lambda \longrightarrow \frac{a\lambda + b}{c\lambda + d} \,. \tag{9.3}$$

The action of the  $SL(2,\mathbb{R})$  symmetry on the 4-form field strength gives us an example of a symmetry of the equations of motion that is not a symmetry of the action. The field strength  $F_{[4]}$  forms an  $SL(2,\mathbb{R})$  doublet together with

$$G_{[4]} = e^{\sigma *} F_{[4]} - \chi F_{[4]} ,$$
 (9.4)

i.e.,

$$\begin{pmatrix} F_{[4]} \\ G_{[4]} \end{pmatrix} \longrightarrow (\Lambda^{\mathrm{T}})^{-1} \begin{pmatrix} F_{[4]} \\ G_{[4]} \end{pmatrix} . \tag{9.5}$$

One may check that these transform the  $F_{[4]}$  field equation

$$\nabla_{M}(e^{\sigma}F^{MNPQ} + \chi^{*}F^{MNPQ}) = 0 \tag{9.6}$$

into the corresponding Bianchi identity,

$$\nabla_{\scriptscriptstyle M}{}^*F^{\scriptscriptstyle MNPQ} = 0 \ . \tag{9.7}$$

Since the field equations may be expressed purely in terms of  $F_{[4]}$ , we have a genuine symmetry of the field equations in the transformation (9.5), but since this transformation cannot be expressed locally in terms of the gauge potential  $A_{[3]}$ , this is not a local symmetry of the action. The transformation (9.3,9.5) is a D=8 analogue of an ordinary Maxwell duality transformation in the presence of scalar fields. Accordingly, we shall refer to the supergravity  $\sigma$ -model symmetries generally as duality symmetries.

The  $F_{[4]}$  field strength of the D=8 theory supports elementary/electric p-brane solutions with p=4-2=2, *i.e.* membranes, which have a d=3 dimensional worldvolume. The corresponding solitonic/magnetic solutions in D=8 have worldvolume dimension  $\tilde{d}=8-3-2=3$  also. So in this case,  $F_{[4]}$  supports both electric and magnetic membranes. It is also possible in this case to have solutions generalizing the purely electric or magnetic solutions that we have considered to solutions that carry both types of charge, *i.e.* dyons.<sup>38</sup>

This possibility is also reflected in the combined Bogomol'ny bound for this situation, which generalizes the single-charge bounds (4.14):

$$\mathcal{E}^2 > e^{-\sigma_{\infty}} (U + \chi_{\infty} V)^2 + e^{\sigma_{\infty}} V^2 , \qquad (9.8)$$

where U and V are the electric and magnetic charges and  $\sigma_{\infty}$  and  $\chi_{\infty}$  are the moduli, *i.e.* the constant asymptotic values of the scalar fields  $\sigma(x)$  and  $\chi(x)^k$ . The bound (9.8) is itself  $\mathrm{SL}(2,\mathbb{R})$  invariant, provided that one transforms in general the moduli  $(\sigma_{\infty}, \chi_{\infty})$  (according to (9.3)) as well as the charges (U, V). For the simple case with  $\sigma_{\infty} = \chi_{\infty} = 0$  that we have mainly considered, the bound (9.8) reduces to  $\mathcal{E}^2 \geq U^2 + V^2$ , which is invariant under an obvious isotropy group  $H = \mathrm{SO}(2)$ .

# 9.3 Charge quantization

So far, we have discussed the structure of p-brane solutions at a purely classical level. At this level, a given supergravity theory can have a continuous spectrum of electrically and magnetically charged solutions with respect to any one of the n-form field strengths that can support such solutions. At the quantum level, however, an important restriction on this spectrum of solutions enters into force. The Dirac-Schwinger-Zwanziger (DSZ) quantization conditions for particles with electric or magnetic charges or for the charges of dyonic particles generalize to p-branes as well. For the simplest case of vanishing moduli, e.g.  $\sigma_{\infty} = \chi_{\infty} = 0$  for our D=8 example, and after suitable normalization, the electric and magnetic charge density numbers (q,p) with respect to a given field strength  $F_{[n]}$  are required to satisfy the relation

$$(qp' - q'p) \in \mathbb{Z} , \qquad (9.9)$$

where (q, p) and (q', p') are the charge density numbers of any two solutions in the spectrum. If, in addition, one assumes the existence of a singly-charged purely electric solution with charge density numbers (1, 0), then the allowed charge density numbers are constrained to lie on an integer charge lattice:  $q, p \in \mathbb{Z}$ .

The quantum-level restriction of allowed charges to a charge lattice has an impact on the allowed symmetry transformations, since electric and magnetic charges are acted upon by supergravity duality symmetries; *c.f.* (9.5). For the simple case of vanishing scalar moduli, this restricts the transformations to

<sup>&</sup>lt;sup>k</sup>In comparing (9.8) to the single-charge bounds (4.14), one should take note that for  $F_{[4]}$  in (9.1) we have  $\Delta = 4$ , so  $2/\sqrt{\Delta} = 1$ .

For the case at hand, 38 one has  $q = \frac{1}{\Omega_4} \int_{\partial M_{5,T}} G$ ,  $p = \frac{1}{\Omega_4} \int_{\partial M_{5,T}} F$ .

those respecting an integer charge lattice. In the D=8 example, this restricts the allowed  $\mathrm{SL}(2,\mathbb{R})$  matrices to be *integer-valued*, thus restricting  $\mathrm{SL}(2,\mathbb{R})$  to  $\mathrm{SL}(2,\mathbb{Z})$ . In the general case, the supergravity duality group ( $\sigma$ -model symmetry group) G given in Table 2 is restricted to  $G(\mathbb{Z})$  in an analogous fashion. In the case of the Cremmer-Julia duality groups in lower dimensions, there is an appropriate definition  $^{40}$  of discretized duality groups like  $E_{7(+7)}(\mathbb{Z})$  as the set of  $\mathrm{SP}(56,\mathbb{Z})$  matrices that preserve the  $E_7$  quadratic invariant.

# 9.4 Counting p-branes

As we have seen at the classical level, the classifying symmetry for solutions in a given scalar vacuum, specified by the values of the scalar moduli, is the linearly-realized isotropy symmetry H given in Table 2. When one takes into account the DSZ quantization condition, this classifying symmetry also becomes restricted to a discrete group, which clearly must be a subgroup of the corresponding  $G(\mathbb{Z})$ , so in general one seeks to identify the group  $G(\mathbb{Z}) \cap H$ . The value of this intersection is modulus-dependent, showing that the homogeneity of the G/H coset space is broken at the quantum level by the quantization condition. Classically, of course, the particular point on the vacuum manifold G/H corresponding to the scalar moduli can be changed by application of a transitively-acting G transformation, for example with group element G. Correspondingly, the isotropy subgroup G0 moves by conjugation with G1.

$$H \longrightarrow qHq^{-1}$$
 (9.10)

The discretized duality group  $G(\mathbb{Z})$  also moves by conjugation, but in the opposite way,

$$G(\mathbb{Z}) \longrightarrow g^{-1}G(\mathbb{Z})g$$
, (9.11)

so the intersection  $G(\mathbb{Z}) \cap H$  takes different values depending on the moduli. For comparison, in ordinary Maxwell theory, one only has a true duality symmetry when the electric charge takes the value unity (in appropriate units), since the duality transformation maps  $e \to e^{-1}$ . Thus, the value e = 1 is a distinguished value.

The distinguished point on the scalar vacuum manifold for general supergravity theories is the one where all scalar moduli vanish. This is the point where  $G(\mathbb{Z}) \cap H$  takes its maximal value. Let us return to our D=8 example to identify what this group is. In that case, for the scalars  $(\sigma, \chi)$ , we may write out the transformation in detail using (9.3):

$$e^{-\sigma} \longrightarrow (d+c\chi)^2 e^{-\sigma} + c^2 e^{\sigma}$$
  
 $\chi e^{-\sigma} \longrightarrow (d+c\chi)(b+a\chi)e^{-\sigma} + ace^{\sigma}$ . (9.12)

Requiring  $a, b, c, d \in \mathbb{Z}$  and also that the point  $\sigma_{\infty} = \chi_{\infty} = 0$  be left invariant, we find only two transformations: the identity and a = d = 0, b = -1, c = 1, which maps

$$e^{-\sigma} \longrightarrow e^{\sigma} + \chi^{2}e^{-\sigma}$$

$$\chi e^{-\sigma} \longrightarrow -\chi e^{-\sigma}. \tag{9.13}$$

Thus, for our truncated system, we find just an  $S_2$  discrete symmetry as the quantum isotropy subgroup of  $SL(2, \mathbb{Z})$  at the distinguished point on the vacuum manifold. This  $S_2$  is the natural analogue of the  $S_2$  symmetry that appears in Maxwell theory when e = 1.

In order to aid in identifying the pattern behind the above D=8 example, suppose that the zero-form gauge potential  $\chi$  is small, and consider the  $S_2$  transformation to lowest order in  $\chi$ . To this order, the transformation just flips the signs of  $\sigma$  and  $\chi$ . Acting on the field strengths  $(F_{[4]}, G_{[4]})$ , one finds

$$(F_{[4]}, G_{[4]}) \longrightarrow (-G_{[4]}, F_{[4]})$$
 (9.14)

One may again check (in fact to all orders, not just to lowest order in  $\chi$ ) that (9.14) maps the field equation for  $F_{[4]}$  into the corresponding Bianchi identity:

$$\nabla_{M}(e^{\sigma}F^{MNPQ} + \chi^{*}F^{MNPQ}) \longrightarrow -\nabla_{M}^{*}F^{MNPQ}. \tag{9.15}$$

Considering the  $S_2$  transformation to lowest order in the zero-form  $\chi$  has the advantage that the sign-flip of  $\phi$  may be "impressed" upon the  $\vec{a}$  dilaton vector for  $F_{[4]}$ :  $\vec{a} \to -\vec{a}$ . The general structure of such  $G(\mathbb{Z}) \cap H$  transformations will be found by considering the impressed action of this group on the dilaton vectors.

Now consider the  $SL(3,\mathbb{R})/SO(3)$  sector of the D=8 scalar manifold, again with the moduli set to the distinguished point on the  $\sigma$ -model manifold. To lowest order in zero-form gauge potentials, the action of  $SL(3,\mathbb{Z}) \cap H$  may similarly by impressed upon the 3-form dilaton vectors, causing in this case a permutation of the  $\vec{a}_i$ , generating for the D=8 case overall the discrete group  $S_3 \times S_2$ . Now that we have a bit more structure to contemplate, we can notice that the  $G(\mathbb{Z}) \cap H$  transformations leave the  $(\vec{a}, \vec{a}_i)$  dot products invariant.

The invariance of the dilaton vectors' dot products prompts one to return to the algebra (6.4) of these dot products and see what else we may recognize in it. Noting that the duality groups given in Table 2 for the higher dimensions D involve  $SL(N,\mathbb{R})$  groups, we recall that the weight vectors  $\vec{h}_i$  of the fundamental representation of  $SL(N,\mathbb{R})$  satisfy

$$\vec{h}_i \cdot \vec{h}_j = \delta_{ij} - \frac{1}{N} , \qquad \sum_{i=1}^N \vec{h}_i = 0 .$$
 (9.16)

These relations are precisely those satisfied by  $\frac{\pm 1}{\sqrt{2}}\vec{a}$  and  $\frac{1}{\sqrt{2}}\vec{a}_i$ , corresponding to the cases N=2 and N=3. This suggests that the action of the maximal  $G(\mathbb{Z}) \cap H$  group (i.e. that for scalar moduli set to the distinguished point on the  $\sigma$ -model manifold) may be identified in general with the symmetry group of the set of fundamental weights for the corresponding supergravity duality group G as given in Table 2. The symmetry group of the fundamental weights is the Weyl group G of G, so the action of the maximal  $G(\mathbb{Z}) \cap H$  G-brane classifying symmetry becomes identified with that of the Weyl group of G.

As one proceeds through the lower-dimensional cases, where the supergravity symmetry groups shown in Table 2 grow in complexity, the above pattern persists:<sup>4</sup> in all cases, the action of the maximal classifying symmetry  $G(\mathbb{Z}) \cap H$  may be identified with the Weyl group of G. This is then the group that *counts* the distinct p-brane solutions<sup>m</sup> of a given mass density (4.1), subject to the DSZ quantization condition and referred to the distinguished point on the manifold of scalar moduli. For example, in D = 7, where from Table 2 one sees that  $G = \mathrm{SL}(5,\mathbb{R})$  and  $H = \mathrm{SO}(5)$ , one finds that the action of  $G(\mathbb{Z}) \cap H$  is equivalent to that of the discrete group  $S_5$ , which is the Weyl group of  $\mathrm{SL}(5,\mathbb{R})$ . In the lower-dimensional cases shown in Table 2, the discrete group  $G(\mathbb{Z}) \cap H$  becomes less familiar, and is most simply described as the Weyl group of G.

Table 3: Examples of p-brane Weyl-group multiplicities

		D								
$F_{[n]}$	Δ	10	9	8	7	6	5	4		
$F_{[4]}$	4	1	1	2						
$F_{[3]}$	4	1	2	3	5	10				
$F_{[2]}$	4	1	1+2	6	10	16	27	56		
	2		2	6	15	40	135	756		
	$^{4}/_{3}$						45	2520		
$F_{[1]}$	4		2	8	20	40	72	126		
	2	1			60	280	1080	3780		
	$\frac{4}{3}$				480	4320	30240 + 2520			

From the above analysis of the Weyl-group duality multiplets, one may tabulate  $^4$  the multiplicities of p-branes residing at each point of the plot given in Figure 4. For supersymmetric p-branes arising from a set of N participat-

 $<sup>^</sup>m$ Of course, these solutions must also fall into supermultiplets with respect to the unbroken supersymmetry; the corresponding supermultiplet structures have been discussed in Ref.<sup>41</sup>

ing field strengths  $F_{[n]}$ , corresponding to  $\Delta=4/N$  for the dilatonic scalar coupling, one finds the multiplicities shown in Table 3. By combining these duality multiplets together with the diagonal and vertical dimensional reduction families discussed in Sections 5 and 7, the full set of  $p \leq (D-3)$  branes shown in Figure 4 becomes "welded" together into one overall symmetrical structure.

# 10 Concluding remarks

In this review, we have focused on the basic structure of p-brane solutions to supergravity theories. Many aspects of this story invite further elaboration. For simplicity, we have concentrated on solutions involving just one independent dilatonic-scalar combination corresponding to the decomposition (6.4); this can, however, be generalized to multi-scalar solutions as shown in Ref. 42 We have also concentrated on fully-isotropic solutions to the transverse-space Laplace equation (2.19); these may be interpreted as single p-brane hyperplanes embedded into the ambient D-dimensional spacetime. This construction also can be generalized, by allowing the harmonic function H(y) to have less than full isotropicity, giving "intersecting p-brane" solutions 43 for which the separation into "worldvolume" and "transverse" directions varies as one moves about at infinity, with only a subset of these directions being "overall worldvolume" or "overall transverse," the rest having a "relative" worldvolume or transverse character. Another aspect of the p-brane story that we have not covered here is the sense in which  $\Delta = 4/N$  solutions for  $N \ge 2$  may be considered to be "bound states" of  $\Delta = 4$  solutions at threshold, i.e. with zero binding energy.  $^{44,41}$  A further generalization of this story is to multiple p-brane solutions linked by branes of lesser dimensionality, <sup>45</sup> a construction which has been termed "brane surgery."

At the classical level that we have confined ourselves to in this review, the singularity structures of p-brane supergravity solutions vary significantly as one moves around the brane-scan of Figure 4. Solutions involving scalar fields are singular at the horizon; extremal solutions without scalars can be continued inside the horizon, either yielding an overall non-singular spacetime such as the D=11 five-brane shown in Figure 3, or leading to timelike interior singularities such as that for the D=11 membrane shown in Figure 1. The fact that all of the non-extremal "black-brane" solutions for  $p\geq 1$  are singular at their inner horizons makes a curious contrast with the non-singular horizons of the extremal cases. Clearly, much remains to be understood about this subject, which may shed further light on the sense in which the extremal BPS p-branes may be considered to be fundamental excitations of the under-

lying quantum theory, an interpretation that would not seem appropriate for the non-extremal cases. If the analogy to the standard Reissner-Nordstrom solution is instructive, it would appear that most of the details of a solution inside the horizon would not have an effect visible in the exterior solution; this may be expected to be encoded in generalizations of the standard "no-hair" theorems for black holes.

When there are timelike singularities in p-brane solutions, one is clearly invited to try to couple in a source. In the present review, we have not engaged in this discussion for reasons of simplicity, but, of course, a considerable amount is known about the structure of such p-brane worldvolume actions.<sup>8,9</sup> The worldvolume actions have been known for some time for those p-branes supported by NS-NS sector field strengths with  $\Delta=4$ ; these constructions follow the pattern of the original D=11 supermembrane action.<sup>19</sup> The main difficulty is to square the p-brane's partial supersymmetry breaking with the original full supersymmetry and Lorentz symmetry of its parent supergravity theory, by the construction of a " $\kappa$ -symmetric" worldvolume action. The  $\kappa$  symmetry achieves an embedding of a partially-nonlinear realisation of supersymmetry into a fully-linear realisation, by the introduction of redundant fermionic gauge degrees of freedom. There is currently much effort being devoted to finding  $\kappa$ -symmetric worldvolume actions for the remaining unsolved cases involving R-R sector antisymmetric-tensor fields.<sup>46</sup>

Of course, the real fascination of this whole subject lies in its connection to emerging understandings in string theory/quantum gravity, and in the possibility of determining some of the structure of that theory by knowledge of its fundamental/solitonic state spectrum, perhaps via an "inverse scattering" analogy to methods that have been very powerful in the study of integrable models.

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